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Allan R. Willms

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A First Course in Complex Analysis

Allan R. Willms

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A First Course in Complex Analysis

Allan R. Willms
University of Guelph

SYNTHESIS LECTURES ON MATHEMATICS AND STATISTICS #45

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ABSTRACT

This book introduces complex analysis and is appropriate for a first course in the subject at typically the third-year University level. It introduces the exponential function very early but does so rigorously. It covers the usual topics of functions, differentiation, analyticity, contour integration, the theorems of Cauchy and their many consequences, Taylor and Laurent series, residue theory, the computation of certain improper real integrals, and a brief introduction to conformal mapping. Throughout the text an emphasis is placed on geometric properties of complex numbers and visualization of complex mappings.

KEYWORDS

complex numbers, analytic functions, contour integration, Cauchy theory, Taylor and Laurent series, residues, conformal mapping

In memory of Dan.

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Preface

This text is the product of course notes I have used to teach an introductory course in complex analysis for many of the past 11 years at the University of Guelph. In developing these notes I have of course benefited from many of the texts that are available on this subject. In particular, I mention the classic text by Churchill and Brown [1990] and the texts by Spiegel et al. [2009] and Saff and Snider [2003].

As with any instructor, I have my own preferences in how to order and present material. How these preferences rate compared to anyone else's is left to the judgment of the reader. One of these ordering preferences is the introduction of the exponential function within the first dozen pages of the text. It is my feeling that the utility of the exponential function for both notational convenience and understanding of the geometry of complex multiplication is simply too great to delay its introduction. Although there are other texts that introduce the exponential relatively early, most that do so provide the reader with little justification, often simply letting Euler's formula be the definition of the exponential of an imaginary number. I have taken a different approach. I assume that students will have seen a real Taylor series and in particular the Taylor series for e^x. It is then completely reasonable from their point of view to simply replace the real quantity x with the complex variable z. After stating a theorem (proved in Chapter 6) that a complex series converges if and only if its real and imaginary parts converge, I rigorously show that the series for e^z converges using familiar convergence tests for real series. Euler's formula then emerges as a consequence of this series definition. Although this approach requires dealing with a series before they are discussed generally, it is, in my opinion, a small price to pay for the benefits returned.

Another feature of the text is an emphasis on the geometric properties of complex numbers and the visualization of functions of a complex variable. To achieve this two types of plots are used: mapping plots, showing regions in the complex plane and their images under given maps, and phase plots, showing the argument of the image of each point, and contours of the modulus. As the anticipated primary mode of use of this text is in an electronic format, I have made liberal use of color in these figures.

What background is the student expected to have? A thorough grounding in single-variable differential and integral calculus is assumed, including an understanding of Taylor series. Familiarity with multi-variable differential calculus, primarily for the understanding of partial derivatives is expected. The student would also benefit from being exposed to multi-variable integral calculus, in particular real line integrals and Green's Theorem, although this is not necessary. The ideas of limits and continuity are expected to be understood, and it is assumed the

student has had exposure to some real analysis, particularly the notions of open and closed sets, although the main points are summarized at the beginning of Chapter 2.

After working through the material in this text, the student should be able to appreciate the references in the following. Of course a familiarity with the original psalm, available in both the Christian and Jewish Bibles, is necessary.

Psalm 23 on the ℂomplex Plane

The Lord is my analytic function; I shall lack no derivatives.
He makes my modulus submaximal on his domain.
He leads my integral to zero around closed contours.
He remakes my parts harmonic.
He guides me on deformable paths, without changing value.

Even though I walk through the valley of the forest of poles,
I will fear no singularity, for you are with me.
Your radius of convergence, and analytic continuation, they comfort me.
You establish a Laurent series before me, surrounding my enemies.
You anoint my argument by 2π; my residues overflow.

Surely continuity and differentiability will follow me on all of my branches,
And I will dwell on a Riemann surface forever.

Allan R. Willms
December 2021

Acknowledgments

I would like to acknowledge the various students of MATH*3260 at the University of Guelph who, through their questions and comments, have improved my teaching and informed my ideas of how to convey the topics in complex analysis. Particularly, I wish to acknowledge the students from the winter of 2021, who suffered through a preliminary version of this text.

Allan R. Willms
December 2021

CHAPTER 1

Basics of Complex Numbers

1.1 INTRODUCTION

The natural numbers are $\mathbb{N} = \{1, 2, 3, \dots\}$. But there is no natural number that is a solution of an equation such as $x + 2 = 1$.

The integers are the set $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. There is an integer solution of $x + 2 = 1$, but there is no integer solution of an equation such as $2x + 3 = 0$.

The rational numbers are the set $\mathbb{Q} = \left\{\frac{a}{b} \ : \ a, b \in \mathbb{Z}, \ b \neq 0\right\}$. There is a rational solution of $2x + 3 = 0$ but there is no rational solution of, for example, $x^2 - 2 = 0$.

The irrational numbers are the set $\overline{\mathbb{Q}}$ consisting of all numbers whose decimal representation does not repeat. There is an irrational solution of $x^2 - 2 = 0$.

The real numbers are the set $\mathbb{R} = \mathbb{Q} \cup \overline{\mathbb{Q}}$. The real numbers are in 1-1 correspondence with points on an infinite line called the real line. However, even using the real numbers there is no solution of some simple polynomial equations such as $x^2 + 2 = 0$. For the solutions of such equations, if they exist, we need a larger set of numbers yet.

An obvious generalization of points on a line is points on a plane. The **Argand Plane** or **Complex Plane** has a horizontal axis, called the **real axis**, which corresponds to the real line, with unit 1, and a vertical axis called the **imaginary axis** with a unit called i. Points z on the Argand plane are complex numbers written in the form $x + iy$ where x and y are real and correspond to the horizontal and vertical distances from the origin $(0, 0)$; see Fig. 1.1. Thus, the complex plane is in one-to-one correspondence with $\mathbb{R}^2$. Specifically, the complex plane is the set

$$\mathbb{C} = \{x + iy \ : \ x, y \in \mathbb{R}\}.$$

We often write yi rather than iy, especially when y is an integer, for example, $z = 3 - 2i$, but this is purely aesthetic. The real part of the complex number $z = x + iy$ is x, and the imaginary part of z is y. These are written as

$$\mathrm{Re}(z) = x \qquad \text{and} \qquad \mathrm{Im}(z) = y.$$

It is important to realize that the imaginary part of a complex number is a *real* number, that is, the imaginary part of $z = x + iy$ is y, not iy. The term "imaginary" is really quite unfortunate as there is nothing imaginary about the number at all. Gauss proposed the term "lateral unit" for i, but unfortunately that never caught on.[1]

[1] Carl Friedrich Gauss (1777–1855), a German mathematician and physicist is considered one of history's most influential mathematicians.

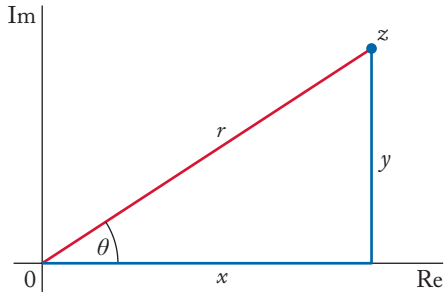


Figure 1.1: The complex plane. Cartesian and polar representations of a complex number z.

It should also be emphasized that there is no ordering of the complex numbers. The statement $z_1 < z_2$ makes no sense for complex numbers z_1 and z_2.

We shall find that not only do complex numbers allow us to theoretically solve all polynomial equations, but complex numbers are extremely useful in many applications. Further, the elementary analysis of differentiable functions of complex variables is extremely elegant and fascinating in its own right.

1.1.1 CARTESIAN AND POLAR FORMS

The form $z = x + iy$ where x and y are real, is called the **Cartesian form** of the complex number. Some texts will also represent a complex number simply using the ordered pair (x, y). Another way to identify a point in the plane is to use the straight line distance, r, from the origin and the angle, θ, between that line and the positive real axis; these quantities are also as shown in Fig. 1.1. The value r is called the **modulus** of the complex number z, and θ is called the **argument**. These are denoted

$$|z| = r \qquad \text{and} \qquad \arg(z) = \theta.$$

Notation convention:

Throughout this textbook we shall use the convention that, unless specifically stated otherwise, the symbols x and y will be reserved for the real and imaginary parts of z, respectively. Similarly, the symbols r and θ will be reserved for the modulus and argument of z, respectively.

It is very important to understand that θ is not unique; any multiple of 2π may be added to θ without changing the point z, since this simply corresponds to circling around the origin an integer number of times. Often we wish to be more precise in our choice of θ, and to do this we could define *any* interval of length 2π (open on one end and closed on the other) and insist that θ lie in this interval. Often we will find it necessary to select a specific interval, say $[-\pi/2, 3\pi/2)$ or $(-3\pi/4, 5\pi/4]$, depending on the problem at hand. However, it is convenient

to have a standard set interval, and if θ lies in this interval we say it is the **Principal Argument** of z and is denoted $\operatorname{Arg}(z)$. We shall adopt the convention that the standard interval is $(-\pi, \pi]$, that is, the principal argument satisfies

$$-\pi < \operatorname{Arg}(z) \leq \pi.$$

This is an obvious choice of interval and is the most common, but the student should be aware that a few texts define the principal argument to lie in the interval $[0, 2\pi)$. The other thing to note is that for $z = 0 + i0$ the argument is undefined.

In summary, the principal argument of a complex number $z \neq 0$ is unique and lies (by our convention) in the interval $(-\pi, \pi]$, while the argument of z may take on this value plus any multiple of 2π, that is,

$$\arg(z) = \operatorname{Arg}(z) + 2\pi k, \qquad k \in \mathbb{Z}.$$

From Fig. 1.1 it is clear that

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta. \tag{1.1}$$

Thus, we may write

$$z = x + iy = r\cos\theta + ir\sin\theta = r\left(\cos\theta + i\sin\theta\right).$$

The quantity $\cos\theta + i\sin\theta$ is often given the short form notation $\operatorname{cis}\theta$. Thus, the **polar form** of a complex number is $z = r\operatorname{cis}\theta$. Many texts use this notation extensively, so you should be familiar with it. However, in Section 1.2 we will introduce a much more convenient notation and so will not use the cis notation beyond that point.

Equation (1.1) expresses the Cartesian form in terms of the polar form, but what about the other way around? By squaring and adding both equations we get $x^2 + y^2 = r^2$, therefore $r = \sqrt{x^2 + y^2}$. We know that we must use the nonnegative square root because r must be nonnegative. By dividing the equations in (1.1) we get $y/x = \tan\theta$. However, isolating for θ is not straightforward. The function $\tan$ is periodic with period π and takes on all real values in each interval of length π, as shown on the left in Fig. 1.2. The inverse function is obtained, as usual, by reflecting in the line $y = x$; this produces the plot on the right in Fig. 1.2. Technically, the inverse tangent, called arctangent and denoted $\arctan$, atan, or sometimes $\tan^{-1}$, is a multivalued function. It has an infinite number of branches, each one defined on the entire real axis but constrained vertically to an interval $(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$, for some $k \in \mathbb{Z}$. The principal branch is the one with $k = 0$ so that it passes through 0; this is the function used by a typical calculator, for example. Every other branch is just a vertical shift of the principal branch by an amount $k\pi$, for some integer k. By general convention, when the notation $\arctan$, atan, or $\tan^{-1}$ is used as a function of a real number, it refers to the principal branch, unless the context indicates that one or more other branches are meant. We shall adopt the same convention when applying this function to a real variable.

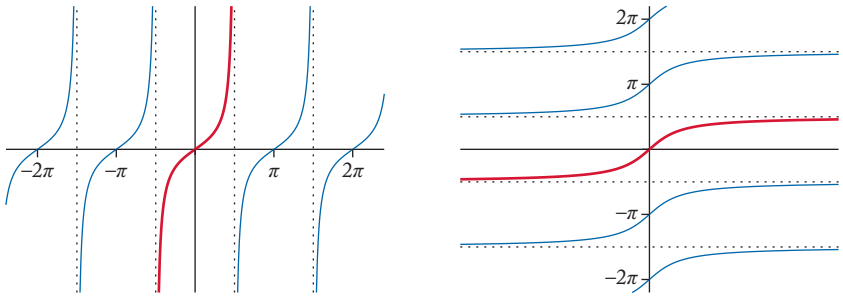


Figure 1.2: The tangent function (left) and its inverse (right). The inverse tangent function has an infinite number of branches. The branch passing through the origin is the principal branch (thick, red) and corresponds to the portion of the tangent function that lies between $-\pi/2$ and $\pi/2$.

The correct branch of $\arctan$ to use to invert the expression $y/x = \tan\theta$ depends on the quadrant in which $z = x + iy$ lies. If z is in the first or fourth quadrant, so x is positive, then the principal argument of z is between $-\pi/2$ and $\pi/2$ and so the principal branch should be used. However, if z is in the third quadrant so x and y are both negative, then the principal argument of z is between $-\pi$ and $-\pi/2$ and so the branch that is $-\pi$ below the principal branch should be used. Similarly, if z is in the second quadrant then the branch that is π above the principal branch should be used. Which branch of $\arctan$ to use is not determined simply by the ratio y/x since both y/x and $(-y)/(-x)$ are the same value but $x + iy$ and $(-x) + i(-y)$ are in different quadrants. For this reason, a "2-input arctangent" function is defined, denoted $\operatorname{atan2}$, which depends on both y and x, not just their ratio. For later convenience, for any given interval J of length 2π, open at one end and closed at the other, we define a general function $\operatorname{atan2}_J$ so that its range is J:

$$\operatorname{atan2}_J(y,x) = \begin{cases} \arctan\left(\frac{y}{x}\right) + (2k+1)\pi \in J, \quad k \in \mathbb{Z}, & \text{if } x < 0, \\ \arctan\left(\frac{y}{x}\right) + 2k\pi \in J, \quad k \in \mathbb{Z}, & \text{if } x \geq 0. \end{cases} \tag{1.2}$$

The effect of the integer k in the above definition is to shift the output value by a multiple of 2π so that the output value is in J. For any given x and y and any given interval J of length 2π, open at one end and closed at the other, there is exactly one integer k that can be selected in (1.2). Note the order of the inputs to this function: y is first, x is second. When $x = 0$ the quantity $\operatorname{atan2}_J(y,0)$ should be interpreted as the limit as x approaches zero; the function is not defined when both x and y are zero. For any positive real number c, $\operatorname{atan2}_J(cy,cx) = \operatorname{atan2}_J(y,x)$. This is only true for *positive* real numbers c, because then the point (cx,cy) is in the same quadrant as (x,y) and the ratio cy/cx is equal to y/x. It is straightforward to verify that $\operatorname{atan2}_J$ has range J, is a continuous function on $\mathbb{R}^2$ except at 0 and along the ray corresponding to the end points of J, where the value jumps by 2π. If $J = (-\pi,\pi]$ then this function yields the

principal argument of $z = x + iy$, that is,

$$\mathrm{Arg}(z) = \mathrm{atan2}_{(-\pi,\pi]}(y, x).$$

Since $J = (-\pi, \pi]$ is the interval most frequently used, we shall adopt the convention that if no subscript is given to atan2 then $J = (-\pi, \pi]$ is assumed. (Many computer languages have a function called atan2 that is precisely this function.) The correct formulas for expressing the modulus and argument of z in terms of x and y are then

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \mathrm{atan2}(y, x) + 2\pi k, \quad k \in \mathbb{Z}. \tag{1.3}$$

One final thing to note, which often comes in useful, is that since $r = \sqrt{x^2 + y^2}$ it follows that

$$\mathrm{Re}(z) \leq |z| \qquad \text{and} \qquad \mathrm{Im}(z) \leq |z|\,.$$

1.1.2 ADDITION AND MULTIPLICATION OF COMPLEX NUMBERS

Now that we have defined complex numbers as points on a plane, we need to generalize the two basic operations of the real numbers, namely addition and multiplication. We use the symbol "+" to denote addition, but, as customary for real variables, we shall typically represent multiplication of two complex variables by simply writing them adjacent to one another. Sometimes we will use the symbol "·" to represent multiplication. As with real variable operations, parentheses are used to prioritize the order of operations or to group expressions. Whatever definitions for addition and multiplication we end up adopting for complex numbers, we want these two operations to satisfy the following properties.

For all z_1, z_2, and $z_3 \in \mathbb{C}$ the following must hold:

1. closed: $z_1 + z_2 \in \mathbb{C}$ and $z_1 z_2 \in \mathbb{C}$;
2. commutative: $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$;
3. associative: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 z_2) z_3 = z_1 (z_2 z_3)$;
4. distributive: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$;
5. identity elements: there exists an additive identity denoted "0" such that $z_1 + 0 = z_1$, and there exists a multiplicative identity denoted "1" such that $z_1 \cdot 1 = z_1$; and
6. unique inverses: for each $z_1 \in \mathbb{C}$ there exists an additive inverse denoted "$-z_1$" such that $z_1 + (-z_1) = 0$, and for each $z_1 \in \mathbb{C}$, $z_1 \neq 0$, there exists a multiplicative inverse denoted $\frac{1}{z_1}$ such that $z_1 \cdot \frac{1}{z_1} = 1$.

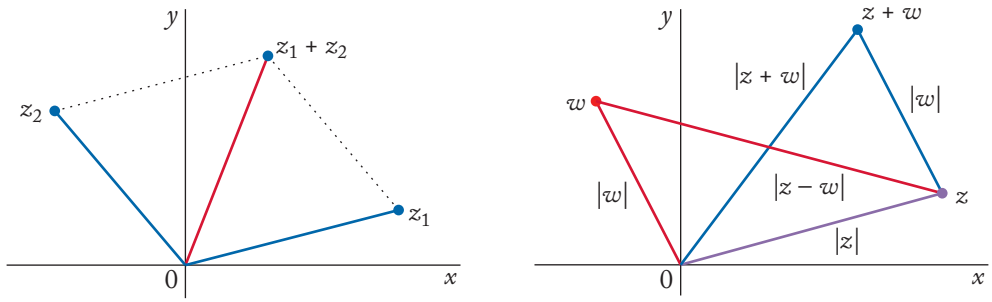


Figure 1.3: Addition of complex numbers (left) acts like vector addition in the plane $\mathbb{R}^2$; real and imaginary parts are added separately. The triangle inequality (right) holds.

Any set of points with operations of addition and multiplication satisfying the above properties is called a **field**. $\mathbb{Q}$ and $\mathbb{R}$ are fields, but $\mathbb{Z}$ is not (which property fails?). We wish to define addition and multiplication of complex numbers so that $\mathbb{C}$ is also a field.

Defining the operation of addition on $\mathbb{C}$ is straightforward, when adding two complex numbers we add the real and imaginary parts separately.

Definition 1.1 Complex Addition. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then the addition of z_1 and z_2 is

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2). \tag{1.4}$$

The geometric notion of this complex addition is the same as that of adding two vectors in the real plane $\mathbb{R}^2$; see Fig. 1.3. It is a straightforward exercise to show algebraically that complex addition defined by (1.4) satisfies the above properties involving only addition. It is also clear from the geometric notion of addition that these properties will be satisfied. For example, when adding three vectors in the plane it does not matter what order you add them in; this is the associativity property. The additive identity is the number $0 + i0$ and the additive inverse of $z = x + iy$ is $-x + i(-y)$, that is, the point obtained by reflecting z through the origin.

This definition of addition also implies that the distance between two points z and w is $|z - w|$, since the vector $z - w$ is the vector one must add to w in order to get z.

Definition 1.2 Distance. The **distance** between two points z and w in the complex plane is the modulus of their difference, $|z - w|$.

The triangle inequality, which is just the geometric statement that the length of one side of a triangle can be no larger than the sum of the lengths of the other two sides, also clearly holds due to our definition of addition. If one considers the triangle with vertices 0, z, and $z + w$ in

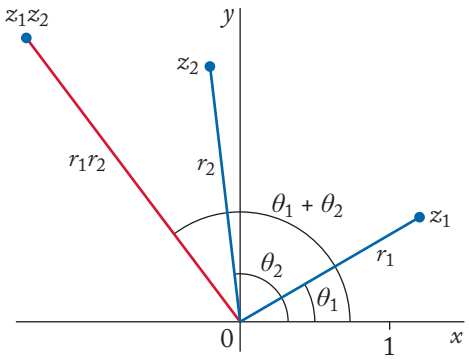


Figure 1.4: Multiplication of complex numbers is a scaling and a rotation.

Fig. 1.3, one gets the familiar form of the triangle inequality: $|z + w| \leq |z| + |w|$. However, if one considers the triangle with vertices 0, w, and z one gets $|z| \leq |w| + |z - w|$, which is usually re-arranged to read: $|z - w| \geq |z| - |w|$. This is an equivalent form of the triangle inequality that is frequently useful.

Unlike addition, it is not immediately clear how one should generalize the operation of multiplication, since there is no notion of multiplying two points in $\mathbb{R}^2$. We want our definition of complex multiplication to coincide with the definition of multiplication of two real numbers in the case that z_1 and z_2 lie on the real axis, that is, if their principal arguments are either 0 or π. How does such real multiplication work? The distance from the origin to the product is the product of the distances from the origin to the two numbers, and the sign of the product is negative if and only if exactly one of the factors is negative. So we want

If z_1 and z_2 are both positive reals:	$z_1z_2 = (r_1 \operatorname{cis} 0)(r_2 \operatorname{cis} 0) = (r_1r_2) \operatorname{cis} 0$
If z_1 is a negative real and z_2 is a positive real:	$z_1z_2 = (r_1 \operatorname{cis} \pi)(r_2 \operatorname{cis} 0) = (r_1r_2) \operatorname{cis} \pi$
If z_1 is a positive real and z_2 is a negative real:	$z_1z_2 = (r_1 \operatorname{cis} 0)(r_2 \operatorname{cis} \pi) = (r_1r_2) \operatorname{cis} \pi$
If z_1 and z_2 are both negative reals:	$z_1z_2 = (r_1 \operatorname{cis} \pi)(r_2 \operatorname{cis} \pi) = (r_1r_2) \operatorname{cis} 0$

Since $\pi + \pi = 2\pi$ is the same angle as 0, it seems clear from the above that a reasonable definition of complex multiplication is that it (real) multiplies the moduli and adds the arguments.

Definition 1.3 Complex Multiplication. If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$ then the multiplication of z_1 and z_2 is

$$z_1z_2 = (r_1r_2) \operatorname{cis} (\theta_1 + \theta_2). \tag{1.5}$$

Multiplication is depicted in Fig. 1.4. The importance of this geometric notion of complex

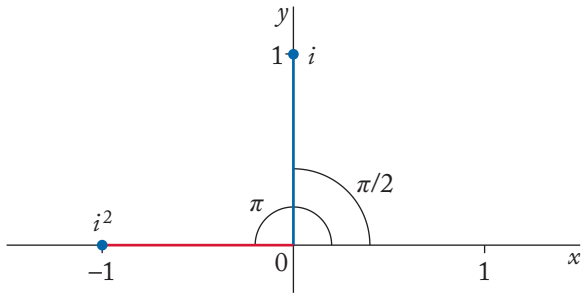


Figure 1.5: $i^2 = -1$

multiplication cannot be overstressed:

Complex multiplication of a number w by a number z = r cis θ consists of scaling the modulus of w by r and rotating w by the angle θ.

What happens if you square a number? By (1.5), if $z = r\,\mathrm{cis}\,\theta$ then $z^2 = r^2\,\mathrm{cis}(2\theta)$, so the modulus squares and the argument doubles. Similarly, $z^3 = r^3\,\mathrm{cis}(3\theta)$ and, in the general case, we get what is known as de Moivre's theorem.[2]

Theorem 1.4 de Moivre's Theorem. *For $z \in \mathbb{C}$ and n a positive integer,*

$$z^n = r^n\,\mathrm{cis}(n\theta).$$

Using these geometric notions of complex addition and multiplication, it is now an easy exercise to verify that the field properties hold. For example, the distributive property is satisfied since adding two vectors then scaling and rotating the result is the same as scaling and rotating the vectors first and then adding. The multiplicative identity is the number $1 + i0 = 1\,\mathrm{cis}\,0$ since it scales by 1 and does not rotate, and the multiplicative inverse of $r\,\mathrm{cis}\,\theta$, $r \neq 0$ is $\frac{1}{r}\,\mathrm{cis}\,(-\theta)$, since the scaling will bring the modulus to 1 and the rotation will bring the argument to zero.

Let us now consider the number $i = 0 + i1 = 1\,\mathrm{cis}\,\frac{\pi}{2}$. This number is one unit directly above the origin. What happens when we square this number? Since the modulus of i is 1, the square will also have modulus 1. Since the arguments add, the argument of the square will be twice the argument of i, hence π. We conclude that $i^2 = 1\,\mathrm{cis}\,\pi = -1$, or, equivalently, $\sqrt{-1} = i$. See Fig. 1.5. We have shown that $i^2 = -1$ is a consequence of our definition of multiplication in the complex plane.

[2] Abraham de Moivre (1667–1754) also worked in the area of probability theory.

It is the author's opinion that minus sign errors are by far the most ubiquitous errors in mathematics. In the subject of complex analysis, these errors are even more prevalent due to the fact that $i^2 = -1$. Be extra vigilant in preventing minus sign errors when doing complex analysis!

The student should also verify that the definition of multiplication, when the numbers are written in Cartesian form, is

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

The above can also be seen to hold because the distributive property holds and each of x_1, iy_1, x_2, and iy_2 can be considered as complex numbers themselves, and because $i^2 = -1$.

1.1.3 EXERCISES

1.1. Sketch the following sets in the complex plane.

(a) $0 < \arg z < \pi/3,\ |z| > 1$, **(b)** $\operatorname{Im} z < 1,\ |\operatorname{Re} z| > 2$, **(c)** $\operatorname{Im} z \leq 2,\ |z| > 2$,
(d) $|3z - 2 + 5i| < 3$, **(e)** $|z - i| \geq |z - 2 - i|$, **(f)** $\operatorname{Arg}(z^2) < \operatorname{Arg}(z)$.

1.2. Write the following in the form $x + iy$, $x, y \in \mathbb{R}$.

(a) $4 - 3i - (2 - i)$, **(b)** $(3 - 2i)(-1 - i)$, **(c)** $i(2 + 5i)(4 + 3i)$.

1.3. If $z = \sqrt{2}\operatorname{cis} \pi/4$, compute z^2 and z^3 and plot them on the complex plane.

1.4. Consider the function $\operatorname{atan2}_J$ defined by (1.2).

(a) Show that $\operatorname{atan2}_J$ has range J and is continuous on $\mathbb{R}^2$ except at the point $(x, y) = (0, 0)$ and along the ray $\theta = a$, where a is one of the end points of J.

(b) Let J_1 and J_2 be two different intervals of length 2π, each closed at one end and open at the other. Let $W = J_1 \cap J_2$. Show that if (x, y) is such that $\operatorname{atan2}_{J_1}(y, x) \in W$ then $\operatorname{atan2}_{J_1}(y, x) = \operatorname{atan2}_{J_2}(y, x)$, and if (x, y) is such that $\operatorname{atan2}_{J_1}(y, x) \notin W$ then $\operatorname{atan2}_{J_1}(y, x)$ and $\operatorname{atan2}_{J_2}(y, x)$ differ by a multiple of 2π.

1.2 THE EXPONENTIAL FUNCTION

Recall that for a real number x, the exponential function is defined as

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

This series converges absolutely[3] for all $x \in \mathbb{R}$ since, by the ratio test,[4] we have

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0.$$

We will now define the exponential function for complex numbers z in exactly the same way:

Definition 1.5 Exponential Function. The exponential function, e^z, for $z \in \mathbb{C}$ is defined as

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

To ensure that this definition is in fact sensible, we will need to verify that the series converges. Convergence is an issue we will discuss in detail in Chapter 6. However, for our purposes here we shall assume that we know what it means for a real series to converge and shall relate that to complex series with the following theorem (which is identical to Theorem 6.9 and proved there).

Theorem 1.6 *If $u_n(z) = a_n(z) + i\,b_n(z)$ and $U(z) = A(z) + iB(z)$ where a_n, b_n, A, and B are real-valued functions of the complex variable z, then $\sum u_n(z)$ converges (uniformly) to $U(z)$ if and only if $\sum a_n(z)$ and $\sum b_n(z)$ converge (uniformly) to $A(z)$ and $B(z)$, respectively.*

In other words, a series of complex numbers converges if and only if the series of its real part and its imaginary part both converge. The "uniformly" adjective essentially refers to the rate of convergence being the same for all z in some region, but that is not material here since we are only concerned at the moment with pointwise convergence of e^z.

By de Moivre's theorem we have that $z^n = r^n \operatorname{cis} n\theta$, where $r = |z|$ and $\theta = \arg(z)$. Therefore the nth term in the series definition of e^z is

$$u_n = \frac{z^n}{n!} = \frac{r^n \operatorname{cis}(n\theta)}{n!} = \frac{r^n\cos(n\theta)}{n!} + i\frac{r^n\sin(n\theta)}{n!}.$$

Now since both $|\cos(n\theta)|$ and $|\sin(n\theta)|$ are bounded above by 1, we have that

$$|\mathrm{Re}(e^z)| = \left|\sum_{n=0}^{\infty}\frac{r^n\cos(n\theta)}{n!}\right| \le \sum_{n=0}^{\infty}\frac{r^n}{n!}, \qquad \text{and} \qquad |\mathrm{Im}(e^z)| = \left|\sum_{n=0}^{\infty}\frac{r^n\sin(n\theta)}{n!}\right| \le \sum_{n=0}^{\infty}\frac{r^n}{n!}.$$

Thus, both the real part and imaginary part are bounded in magnitude by the series $\sum_{n=0}^{\infty}\frac{r^n}{n!}$, which clearly converges since it is by definition the real series for e^r. By the comparison test then,

[3] Absolute convergence means that the series obtained by replacing all terms with their absolute values converges.
[4] We will look more closely at the ratio test and other tests for convergence of series in Section 6.1.

both the real part and imaginary part of e^z are absolutely convergent for each z. We conclude that the series for e^z converges absolutely for every $z \in \mathbb{C}$.

1.2.1 EULER'S FORMULA

Now consider the special case where $z = iy$, $y \in \mathbb{R}$. In this case we have

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \frac{1}{0!} + i\frac{y}{1!} - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} + \dots,$$

where we have used that fact that $i^2 = -1$, $i^3 = -i$, etc. Note that the even terms are real while the odd terms are purely imaginary. Since we know the series is absolutely convergent, we may re-arrange the order of summation, collecting the even and odd terms together, to get

$$e^{iy} = \sum_{m=0}^{\infty} \frac{(-1)^m y^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} \frac{(-1)^m y^{2m+1}}{(2m+1)!}.$$

You may recognize these two series. They are in fact the Taylor series for $\cos y$ and $\sin y$, which can be verified by checking any introductory calculus textbook. We have therefore arrived at **Euler's formula**, which is often written with the variable θ rather than y:

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \text{for } \theta \in \mathbb{R}.$$

For $\theta = \pi$, Euler's formula provides what by some is considered the most elegant equation in all of mathematics:

$$e^{i\pi} + 1 = 0.$$

In this formula we have the most elementary of all numbers, 1, the exceedingly important concept of zero, the most basic arithmetic operation of addition and relational operation of equality, the transcendental number e, special due to the fact that the function e^x has derivative equal to its value at each x, the geometric constant π, which is the ratio of the circumference to the diameter of a circle, and the fundamental complex number i being the square root of -1. That all these things can be put together in the above simple form to produce a true statement is a thing of mathematical beauty.

1.2.2 THE EXPONENTIAL AS POLAR FORM

Using Euler's formula, we see that the notation $\operatorname{cis}\theta$ is simply equal to $e^{i\theta}$. We may now write the polar form of a complex number using the exponential:

$$z = re^{i\theta}, \qquad \text{where} \quad r = |z| \text{ and } \theta = \arg z.$$

This notation is much more convenient than the form $r\operatorname{cis}\theta$, hence from now on we will consider $z = re^{i\theta}$ as the polar form of a complex number. Some texts call this the "exponential form" of a complex number.

Part of the reason that this notation is convenient is because it immediately suggests how complex multiplication works. For example, if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1+\theta_2)}.$$

Thus, the modulus of the product is the product of the moduli of the factors, and the argument of the product is the sum of the arguments of the factors. And, de Moivre's theorem in this notation becomes

$$z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}.$$

In the above, we have assumed that the property of the real exponential function that says $e^{a+b} = e^a e^b$ for $a, b \in \mathbb{R}$ also holds if a and b are complex. This seems reasonable but it should be verified. Let a and b be any two complex numbers, then, from the definition of the exponential and using the binomial theorem (recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$) we have

$$\begin{aligned} e^{a+b} &= \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a^{n-k}}{(n-k)!} \frac{b^k}{k!} \\ &= \underbrace{\frac{a^0 b^0}{0!0!}}_{n=0} + \underbrace{\frac{a^1 b^0}{1!0!} + \frac{a^0 b^1}{0!1!}}_{n=1} + \underbrace{\frac{a^2 b^0}{2!0!} + \frac{a^1 b^1}{1!1!} + \frac{a^0 b^2}{0!2!}}_{n=2} \\ &\quad + \underbrace{\frac{a^3 b^0}{3!0!} + \frac{a^2 b^1}{2!1!} + \frac{a^1 b^2}{1!2!} + \frac{a^0 b^3}{0!3!}}_{n=3} + \underbrace{\cdots}_{n=\cdots}. \end{aligned}$$

For every pair of nonnegative integers k and m, the sum on the right-hand side of the above expression contains exactly one term of the form $\frac{a^m}{m!}\frac{b^k}{k!}$. Also, since e^{a+b} is absolutely convergent, we may re-arrange the sum in any order without affecting the result. Thus, we may write the sum as

$$e^{a+b} = \left(\sum_{m=0}^{\infty} \frac{a^m}{m!}\right)\left(\sum_{k=0}^{\infty} \frac{b^k}{k!}\right) = e^a e^b,$$

because the product of the two series in the above expression also has exactly one term of the form $\frac{a^m}{m!}\frac{b^k}{k!}$ for each pair of nonnegative integers k and m. We have proved that indeed this property of the exponential function is valid for complex variables as well.

From this property of the exponential we also have

$$e^z = e^{x+iy} = e^x e^{iy},$$

therefore the modulus of e^z is e^x and the argument of e^z is y. Further, the exponential is a periodic function with period $2\pi i$ since

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z \left(\cos(2\pi) + i \sin(2\pi)\right) = e^z.$$

A further useful property of the exponential is that the set of points $e^{i\theta}$, $0 \le \theta \le 2\pi$, traces out the unit circle centered at the origin in the complex plane.

1.2.3 EXERCISES

1.5. Let x_0 be a fixed real number and consider the vertical line with points $z = x_0 + iy$, $y \in \mathbb{R}$. What set of points is obtained when the exponential function is applied to all points on the line? Answer the same question for the horizontal line $z = x + iy_0$.

1.6. Show that $e^{i\pi/2} = i$.

1.7. If $z = \sqrt{3}\, e^{-i\pi/6}$, compute z^3.

1.8. If $z = e^{i\pi/3}$, is there a value of n such that $z = z^n$?

1.3 CONVERSION BETWEEN CARTESIAN AND POLAR FORMS

In Section 1.1.1 we developed the formulas for converting between Cartesian and polar forms of a complex number:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \Longleftrightarrow \quad r = \sqrt{x^2 + y^2}, \quad \theta = \operatorname{atan2}(y, x) + 2\pi k, \quad k \in \mathbb{Z}.$$

For the majority of angles, θ, the values of $\cos\theta$ and $\sin\theta$ are not simple expressions, and conversely for most values of x and y the value of $\operatorname{atan2}(y, x)$ is not a simple expression. However, there are a few special angles where there is a nice, simple relationship, for example, $\theta = \pi/2$. Since these special angles are used extensively in examples, it is worthwhile reviewing the points on the unit circle corresponding to these angles.

Recall that the x and y coordinates of a point on the unit circle are equal to $\cos\theta$ and $\sin\theta$, respectively, where θ is the angle between the positive x-axis and the line joining the point to the origin. See Fig. 1.6. The special angles in the first quadrant are 0, $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$. Adding or subtracting multiples of $\pi/2$ give the special angles in all four of the quadrants. To memorize the values of $\cos\theta$ and $\sin\theta$ for these special angles one needs only be able to count from 0 to 4, apply a square root, and divide by two. For example, starting at the angle 0, the y-coordinate is clearly 0. Thus, counting from 0 to 4, taking a square root and dividing by two for each number yields the sequence

$$0, \ \frac{1}{2}, \ \frac{\sqrt{2}}{2}, \ \frac{\sqrt{3}}{2}, \ 1. \tag{1.6}$$

These are the values of $\sin\theta$ for the angles 0, $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$, in order. Similarly, starting at the angle $\pi/2$, where the x-coordinate is clearly 0, the same sequence (1.6) gives the values of $\cos\theta$ for the angles $\pi/2$, $\pi/3$, $\pi/4$, $\pi/6$, and 0, in order. Similar reasoning can be used to obtain

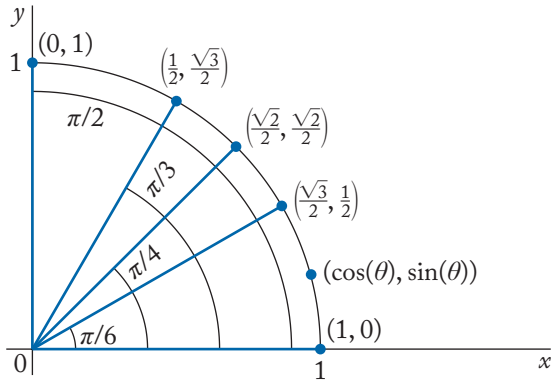


Figure 1.6: Special angles. The coordinates of points on the unit circle are given by $(\cos\theta, \sin\theta)$. These values for special angles in the first quadrant are given. Cosine and sine values of special angles in the other quadrants are obtained by rotation and reflection.

the values of $\cos\theta$ and $\sin\theta$ for the special angles in the other quadrants, or one can simply think of reflections or rotations that map the angle of interest to the first quadrant.

Example 1.7 Write each of i, $3+3i$, $2-i2\sqrt{3}$, and $-3-i\sqrt{3}$ in polar form.

For $z=i$, the argument is clearly $\pi/2$ and the modulus is 1, so $z=e^{i\pi/2}$.

For $z=3+3i$, since $x=y$, the argument is clearly $\pi/4$. The modulus is $r=\sqrt{3^2+3^2}=3\sqrt{2}$, so $z=3\sqrt{2}\,e^{i\pi/4}$.

For $z=2-i2\sqrt{3}$ the modulus is $r=\sqrt{2^2+(2\sqrt{3})^2}=\sqrt{16}=4$. Factoring out the modulus we have

$$z=2-i2\sqrt{3}=4\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right).$$

The angle on the unit circle where $\cos\theta=1/2$ and $\sin\theta=-\sqrt{3}/2$ is $\theta=-\pi/3$ so $z=4e^{-i\pi/3}$.

For $z=-3-i\sqrt{3}$ the modulus is $r=\sqrt{3^2+3}=2\sqrt{3}$. Factoring out the modulus we have

$$z=-3-i\sqrt{3}=2\sqrt{3}\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right).$$

The angle on the unit circle where $\cos\theta=-\sqrt{3}/2$ and $\sin\theta=-1/2$ is $\theta=-5\pi/6$ so $z=2\sqrt{3}\,e^{-i5\pi/6}$.

Conversion between different forms of a complex number is often required because addition is most easily performed when the numbers are in Cartesian form, while multiplication is most easily performed when the numbers are in polar form.

1.3.1 EXERCISES

1.9. **(a)** Write $2e^{i5\pi/4}$ in the form $x + iy$, $x, y \in \mathbb{R}$.

(b) Write $-2 + 2i$ in the form $re^{i\theta}$, $r > 0$, $-\pi < \theta \leq \pi$.

1.10. Write the following in Cartesian form. (In many cases it may be easier to change to polar form before doing multiplications.)

(a) $\left(-\sqrt{3} - i\right)^7$, **(b)** $4e^{i2\pi/3} + \left(1 + \sqrt{3}\,i\right)^4$, **(c)** $(2 + 3i)e^{i\pi/4}$,
(d) $\left(1 + i\right)^3 \left(\sqrt{3} + i\right)^3$, **(e)** $\left(\sqrt{2} + i\sqrt{2}\right)^3 + \left(\sqrt{2}e^{i\pi/3}\right)^4$.

1.11. Use the binomial theorem and de Moivre's theorem to express $\cos n\theta$ and $\sin n\theta$ as sums of terms of the form $\cos^{n-k}\theta \sin^k \theta$, where $0 \leq k \leq n$. Give explicit expressions for the cases $n = 3$ and $n = 4$.

1.4 CONJUGATION

Definition 1.8 Complex Conjugate. The conjugate of a complex number $z = x + iy = re^{i\theta}$ is denoted $\overline{z}$ and is defined as

$$\overline{z} = x - iy = re^{-i\theta}.$$

Thus, the geometric operation of conjugation is reflection across the real axis. Algebraically, it negates the imaginary part, or equivalently negates the argument. Using the geometric notions of conjugation, addition, and multiplication it should be easy to see that

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \qquad \text{and} \qquad \overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}.$$

From the definition of conjugation, it immediately follows that

$$z\overline{z} = re^{i\theta} \cdot re^{-i\theta} = r^2.$$

Thus, multiplying a number by its conjugate results in the square of the modulus. Using this fact it is now easy to find the multiplicative inverse of a number z in Cartesian form:

$$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{r^2} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

This algebraic trick of multiplying the numerator and denominator of a rational expression of complex numbers by the conjugate of the denominator is a very useful way of simplifying complex expressions. Of particular note is that

$$\frac{1}{i} = \frac{-i}{i(-i)} = -i.$$

Example 1.9 Write the expression $w = \dfrac{2-3i}{1+2i}$ in Cartesian form.

$$w = \frac{2-3i}{1+2i} = \frac{(2-3i)(1-2i)}{(1+2i)(1-2i)} = \frac{2-6-4i-3i}{1^2+2^2} = \frac{-4-7i}{5} = -\frac{4}{5} - i\frac{7}{5}.$$

1.4.1 EXERCISES

1.12. If $z = \overline{z}$ what can you say about z?

1.13. Using the Cartesian form of z, show explicitly that $z\overline{z} = |z|^2$.

1.14. If $\operatorname{Arg}(z) > 0$ what can you say about $\operatorname{Arg}(1/z)$?

1.15. Find all z that satisfy $\overline{z} = 1/z$.

1.16. Write $z = \dfrac{5-2i}{2+i}$ in Cartesian form. What is $|z|$?

1.17. **(a)** Using the polar form of z verify that $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$.

(b) Using the Cartesian form of z verify that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

(c) Verify that $\dfrac{z}{\overline{z}} = e^{i2\arg(z)}$.

1.5 INTEGER AND RATIONAL POWERS

For a positive integer n we already know that

$$z^n = r^n e^{in\theta}.$$

We define negative integer powers as follows. For $n > 0$,

$$z^{-n} = \frac{1}{z^n} = \frac{1}{r^n e^{in\theta}} = r^{-n} e^{-in\theta}.$$

Thus, de Moivre's theorem holds for all integers n, where we define $z^0 = 1$.

What about rational powers of z? First consider $z^{1/n}$ where $n \in \mathbb{Z}$, $n \neq 0$. The obvious way to proceed is to simply raise the polar form of z to the power $1/n$:

$$z^{1/n} = (re^{i\theta})^{1/n} = r^{1/n} e^{i\theta/n}.$$

However, it is necessary to remember that the argument θ of any complex number is not unique; we may add any multiple of 2π to it. Thus, the correct definition of $z^{1/n}$ is as follows.

Definition 1.10 Roots of a complex number. If $z = re^{i\theta}$ then the n values of $z^{1/n}$ are

$$z^{1/n} = \left(re^{i(\theta+2\pi k)}\right)^{1/n} = r^{\frac{1}{n}} e^{i(\theta+2\pi k)/n}, \qquad 0 \le k < n.$$

It does not matter which particular value of θ is chosen for the argument of z because shifting it by a multiple of 2π is the same as changing the value of k. The reason we only need to consider the integers $0 \le k < n$ and not all integers is because other values of k will simply yield an argument that is a factor of 2π different than these. For example, $k = n$ gives an argument for $z^{1/n}$ that is 2π larger than the argument obtained with $k = 0$. Thus, we see that there are exactly n nth roots of a complex number. Further, these roots are evenly spaced around a circle of radius $r^{1/n}$. With our geometric notion of multiplication, which scales the modulus and adds the argument, we can see that the modulus of the nth roots must be $r^{1/n}$, so that when you multiply n copies of it together you get r, and the arguments of the nth roots need to be θ/n plus a multiple of $2\pi/n$ so that when you add n copies of it together you get θ plus a multiple of 2π.

The fact that there are multiple values of $z^{1/n}$ should come as no surprise. Already with real numbers you are familiar with the fact that there are two square roots of any positive number, one of them being positive and the other negative. These correspond to the integers $k = 0$ and $k = 1$, respectively. For example,

$$9^{1/2} = \left(9e^{i0}\right)^{1/2} = 9^{1/2}e^{i(0+2\pi k)/2} = \begin{cases} 3e^{i0/2} = 3 & \text{if } k = 0, \\ 3e^{i(0+2\pi)/2} = 3e^{i\pi} = -3 & \text{if } k = 1. \end{cases}$$

The principal nth root of z is defined using the principal argument.

Definition 1.11 The **principal nth root of z** is $|z|^{1/n}\, e^{i\,\mathrm{Arg}(z)/n}$, where $\mathrm{Arg}(z)$ is the principal argument of z.

Since the principal argument of a positive real number is 0, it follows that the principal nth root of a positive real number is also a positive real number. So, the principal square root of 9 is 3, and the principal cube root of 64 is 4, etc.

Example 1.12 Compute and draw on the complex plane the values of $z^{1/3}$, where (a) $z = 8$, and (b) $z = i\frac{27}{8}$. Write these values in both polar and Cartesian form. Also indicate which root is the principal root.

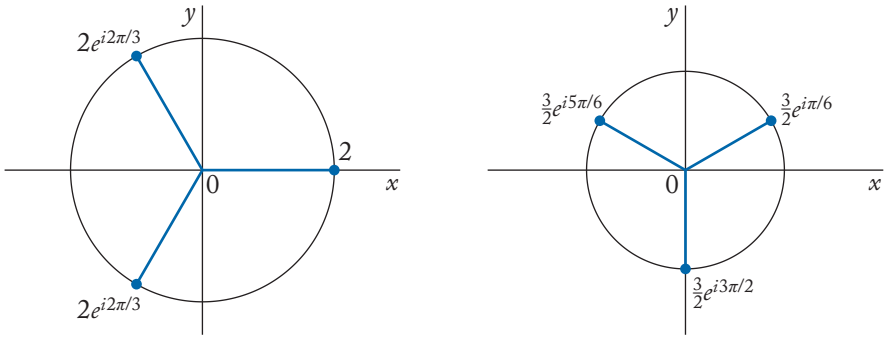


Figure 1.7: Solutions for Example 1.12. Left, (a), cube roots of $z = 8$. Right, (b), cube roots of $z = i27/8$.

(a) We have that $z = 8e^{i0}$, therefore

$$\begin{aligned} z^{1/3} &= 2e^{i(0+2\pi k)/3}, \qquad k = 0, 1, 2 \\ &= 2e^{i0}, \quad 2e^{i2\pi/3}, \quad 2e^{i4\pi/3}, \qquad \text{(polar form)}, \\ &= 2, \quad -1 + i\sqrt{3}, \quad -1 - i\sqrt{3}, \qquad \text{(Cartesian form)}. \end{aligned}$$

Since $\mathrm{Arg}(z) = 0$, the principal cube root is 2. These values are sketched in the left plot of Fig. 1.7.

(b) We have that $z = \frac{27}{8}\, e^{i\pi/2}$, therefore

$$\begin{aligned} z^{1/3} &= \frac{3}{2}\, e^{i(\frac{\pi}{2}+2\pi k)/3}, \qquad k = 0, 1, 2 \\ &= \frac{3}{2}\, e^{i\pi/6}, \quad \frac{3}{2}\, e^{i5\pi/6}, \quad \frac{3}{2}\, e^{i3\pi/2}, \qquad \text{(polar form)}, \\ &= \frac{3\sqrt{3}}{4} + i\frac{3}{4}, \quad -\frac{3\sqrt{3}}{4} + i\frac{3}{4}, \quad -i\frac{3}{2}, \qquad \text{(Cartesian form)}. \end{aligned}$$

Since $\mathrm{Arg}(z) = \pi/2$, the principal cube root is $\frac{3}{2}\, e^{i\pi/6}$. These values are sketched in the right plot of Fig. 1.7.

The nth roots of unity are solutions of $z = 1^{1/n}$. These are the n points on the unit circle equally spaced starting at 1. Figure 1.8 shows these roots for the cases $n = 4$, $n = 5$, and $n = 6$.

Consider now a rational power of the form $z^{p/q}$ where p and q are integers, $q \neq 0$. This can be computed either as $(z^p)^{1/q}$ or as $(z^{1/q})^p$, provided one first writes z including all of its

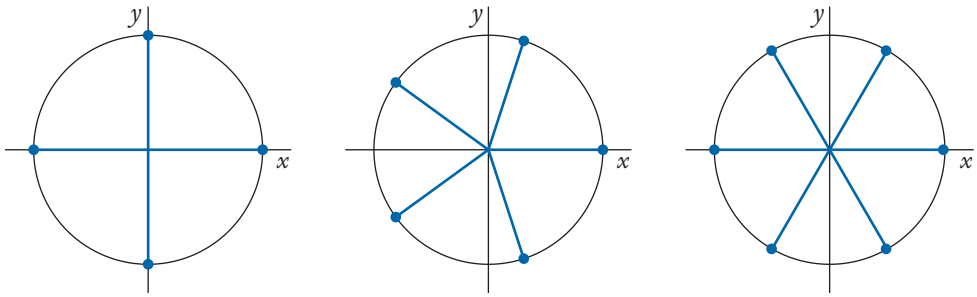


Figure 1.8: The nth roots of unity, for (left to right) $n = 4, 5, 6$.

possible arguments. For example,

$$(-8)^{2/3} = (8e^{i(\pi+2\pi k)})^{2/3} = (64e^{i2(\pi+2\pi k)})^{1/3} = 4e^{i2(\pi+2\pi k)/3}, \qquad k \in \mathbb{Z}.$$

or equivalently,

$$(-8)^{2/3} = (8e^{i(\pi+2\pi k)})^{2/3} = (2e^{i(\pi+2\pi k)/3})^2 = 4e^{i2(\pi+2\pi k)/3}, \qquad k \in \mathbb{Z}.$$

In both cases you get the three distinct solutions: $4e^{i2\pi/3}$, 4 and $4e^{-i2\pi/3}$, corresponding to choosing $k = 0, 1$, or 2. As a small word of warning, you should be sure to write z in terms of all its valid arguments prior to taking the power p or the power $1/q$. If you take the power p first and then write that result in terms of all its valid arguments and then take the power $1/q$, you will get extraneous answers in the case that p and q have a common factor. For example, if you were asked to compute $i^{2/4}$ you might be tempted to first say, $i^2 = -1$ so the result is $(-1)^{1/4}$, which would be

$$(-1)^{1/4} = (e^{i(\pi+2\pi k)})^{1/4} = e^{i(\pi+2\pi k)/4} = e^{i\pi/4}, \quad e^{i3\pi/4}, \quad e^{i5\pi/4}, \quad e^{i7\pi/4}.$$

However, there must be only two solutions because $i^{2/4} = i^{1/2}$, and we know that there are just two second roots of any number. Of these four solutions only $e^{i\pi/4}$ and $e^{i5\pi/4}$ are valid values for $i^{1/2}$, since when you square these two you get i but when you square the other two you get $-i$. So where did the other two solutions come from? What was our error? The error was stating that $e^{i(\pi+2\pi k)}$ is a valid representation of $-1 = i^2$ (which it is), but then implicitly assuming that $e^{i(\pi+2\pi k)/2}$ is a valid representation of i (which it is not for all k). The valid representations of i are $e^{i(\pi/2+2\pi k)} = e^{i(\pi+4\pi k)/2}$. This error can be avoided by making sure you first write z in terms of all its valid arguments before raising it to the power p/q. For this example we would compute as

$$\begin{aligned} i^{2/4} &= (e^{i(\pi/2+2\pi k)})^{2/4} = e^{i2(\pi/2+2\pi k)/4} = e^{i(\pi/2+2\pi k)/2}, \qquad k \in \mathbb{Z}, \\ &= e^{i\pi/4}, \quad e^{i5\pi/4}. \end{aligned}$$

The error can also be avoided by cancelling all common factors in p and q prior to computing $z^{p/q}$.

In summary, if p and q are nonzero integers with no common factors, then there are q distinct values of $z^{p/q}$ given by

$$z^{p/q} = |z|^{p/q}\, e^{i(\operatorname{Arg}(z)+2\pi k)p/q}, \qquad k = 0, 1, \ldots, q-1.$$

The **principal value** of $z^{p/q}$ is the value obtained by selecting the principal qth root in the above, that is,

$$z^{p/q} = |z|^{p/q}\, e^{i \operatorname{Arg}(z) p/q}.$$

1.5.1 EXERCISES

1.18. Write $\left(\dfrac{-\sqrt{3}+i}{3e^{i\pi/3}}\right)^{-5}$ in the form $z = re^{i\theta}$, with $-\pi < \theta \le \pi$.

1.19. For each of the following, determine all values in Cartesian form and plot the solutions on the complex plane.

(a) $(9i)^{1/2}$, **(b)** $(1+i)^{1/3}$, **(c)** $(-16i)^{10/8}$, **(d)** $(1-i)^{6/9}$.

1.20. For a given $z \in \mathbb{C}$, define w to be the principal value of $z^{5/4}$. Find any specific complex number z such that the principal value of $w^{4/5}$ is not equal to z.

1.21. Let n be a positive integer. Show that the product of the n values of $z^{1/n}$ is z if n is odd and is $-z$ if n is even.

1.22. Consider the quadratic equation $az^2 + bz + c = 0$, where a, b, and c are complex constants, $a \neq 0$.

(a) Show, in the way described below, that the quadratic formula

$$\frac{-b + \left(b^2 - 4ac\right)^{1/2}}{2a} \tag{1.7}$$

gives the two roots of the quadratic equation. Let $b^2 - 4ac = \rho e^{i\phi}$; then $\left(b^2 - 4ac\right)^{1/2}$ has two values: $\sqrt{\rho}\, e^{i\phi/2}$ and $\sqrt{\rho}\, e^{i(\phi+2\pi)/2}$. Now compute $(z - z_1)(z - z_2)$ where z_1 and z_2 are the two values given by (1.7).

(b) Use the result of (a) to find all solutions of

(i) $z^2 + iz + 1 = 0$, (ii) $z^2 + z - i\sqrt{3}/4 = 0$,

(iii) $iz^2 + 2iz + 2 + i = 0$, (iv) $z^4 + z^2 + 1 = 0$,

giving your answers in the form $x + iy$.

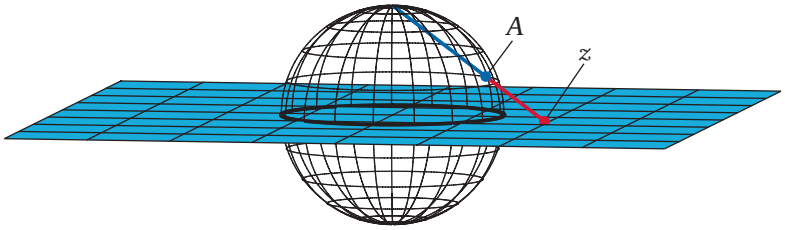


Figure 1.9: Stereographic projection of the complex plane onto the unit sphere.

1.6 STEREOGRAPHIC PROJECTION

Consider the complex plane and add a vertical direction. Place a unit sphere, S^2, at the origin so that the equator of the sphere intersects the complex plane on the unit circle, as shown in Fig. 1.9. The north pole of the sphere sits directly above the origin. For any point $z \in \mathbb{C}$ define a mapping Φ from $\mathbb{C}$ to S^2 by drawing a straight line between the north pole of the sphere and z. This line necessarily intersects the unit sphere at the north pole and at exactly one other point A. The mapping is defined as $\Phi(z) = A$ and is called **stereographic projection**. Actually, this term is used in both directions; we say the stereographic projection of z is A, and we also say the stereographic projection of A is z, under the inverse mapping Φ^{-1}. The sphere is called the **Riemann sphere**.[5]

Several properties of stereographic projection are immediately evident from the figure.

1. The origin maps to the south pole of the sphere.
2. The unit circle on $\mathbb{C}$ maps to the equator of the sphere.
3. Points inside the unit circle map to the southern hemisphere and points outside the unit circle map to the northern hemisphere.
4. A circle centered at the origin maps to a circle of constant latitude on the sphere.
5. A ray from the origin maps to a curve of constant longitude on the sphere (half a great circle).
6. A line on the plane maps to a circle on the sphere. (This is because connecting the north pole to every point on the line sweeps out a plane, and we know planes must intersect a sphere in a circle.)
7. As the magnitude of z gets large, its image approaches the north pole.

[5]Georg Friedrich Bernhard Riemann (1826–1866) was a German mathematician who made great advances in analytic number theory, differential geometry, and other areas. He is perhaps best known for his rigorous definition of the Riemann integral. His primary contribution to complex analysis is the idea of Riemann surfaces, which appear in Sec. 3.6.

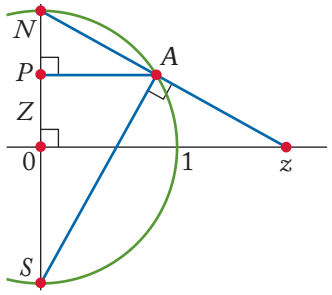


Figure 1.10: Stereographic projection mapping of z to the Riemann sphere. Z is the distance from 0 to P.

The only point on the sphere that is not an image of some point on the complex plane is the north pole itself. Due to the last property listed above, we can say that the north pole corresponds to the **point at infinity**. This point at infinity is the same regardless of the argument; no matter which direction one moves away from the origin on the complex plane, the stereographic image of the point will approach the north pole of the sphere. Thus, there is only *one* point at infinity. Stereographic projection allows one to think of complex numbers as points on the Riemann sphere rather than points on a plane, and it is this notion of a single point at infinity that is the most important aspect of stereographic projection. The complex plane together with the point at infinity is called the **extended complex plane**.

It is a moderately difficult geometry exercise to work out the exact formulas for $\Phi(z)$ in terms of either coordinates (x_1, x_2, x_3) on S^2 with $x_1^2 + x_2^2 + x_3^2 = 1$, or in terms of latitude and longitude coordinates; see Problem 1.23.

One other fascinating thing about stereographic projection is that it maps all lines and circles on the plane to circles on the sphere. We have already noted above that this is true for any line and is true for circles that are centered at the origin. It is a bit more difficult to show that this statement is true for more general circles, but we shall be able show in Section 2.3.1 that it is true for circles that pass through the origin. A nice proof of the general result using inversion in spheres is given in Needham [1997].

Some authors define stereographic projection by placing the sphere above the plane so that the south pole is at the origin. In this case, if the sphere has radius $1/2$ then the unit circle will map to the equator. The geometry of this alternate form of projection is qualitatively the same.

1.6.1 EXERCISES

1.23. Let (x, y, z) be a three-dimensional coordinate system aligned so that the positive x- and y-axes are the positive real and imaginary axes of $\mathbb{C}$, respectively. Find explicit formulas for $(X, Y, Z) = \Phi(z)$ in terms of x and y, where $z = x + iy$. [Hints: First

note that the direction θ is not altered by the projection so that $X + iY = K(x + iy)$ for some positive real constant K. Consider Fig. 1.10. There are three unknowns, $|PA|$, $|NA|$, and Z. It is possible to obtain all of these in terms of $|z|$ using geometry. One way to do this is to get one relation from the similar triangles ONz and PNA, one from the similar triangles ANS and PNA, and one from the Pythagorean theorem applied to PNA. Once you have $|PA|$ you can compute K, since $|PA| = \sqrt{X^2 + Y^2}$.]

CHAPTER 2

Functions of a Complex Variable

2.1 SET TERMINOLOGY

Here we list some terminology for sets in $\mathbb{C}$ that we will encounter throughout the remainder of the book. Let S be a set of complex numbers.

1. **Empty set.** The set containing no points is called the empty set and is denoted $\emptyset$.

2. **Complement.** The complement of a set S is the set $\tilde{S}$ consisting of all points not in S.

3. **Neighborhood.** A neighborhood of a point z_0 is simply all the points nearby, specifically the set
$$N_\delta(z_0) = \{z \in \mathbb{C} \,:\, |z - z_0| < \delta\}, \qquad \delta > 0,$$
is the "δ neighborhood" of z_0. A "deleted neighborhood" of z_0 is one that excludes the point z_0 itself, hence the set condition is replaced with $0 < |z - z_0| < \delta$. A deleted neighborhood of z_0 shall be denoted $N_\delta(z_0) \setminus \{z_0\}$.

4. **Limit Points/Cluster Points/Accumulation Points.** Limit points of a set S are points that have other points of S close by, no matter how close you require them to be.

   The point z_0 is a limit point of a set S if every deleted neighborhood of z_0 contains some points in S, that is,
$$(N_\delta(z_0) \setminus \{z_0\}) \cap S \neq \emptyset, \qquad \forall\, \delta > 0.$$

   The point z_0 itself need not be in S. For example, if S is the set of points inside but *not* on the unit circle, then the limit points of S are all points inside *or* on the unit circle. If S is a discrete set, for example, all the integers, then it has no limit points since, for any z_0 one can always find a sufficiently small δ so that $(N_\delta(z_0) \setminus \{z_0\}) \cap \{z_0\}$ contains no integers.

5. **Interior points.** These are points completely surrounded by points of S. The point z_0 is an interior point of the set S if there exists a $\delta > 0$ such that $N_\delta(z_0) \subset S$.

6. **Boundary points.** These are points that have both points inside and outside S that are nearby.

The point z_0 is a boundary point of the set S if for every $\delta > 0$

$$N_\delta(z_0) \cap S \neq \emptyset \qquad \text{and} \qquad N_\delta(z_0) \cap \tilde{S} \neq \emptyset.$$

Equivalently, boundary points are limit points that are not interior points.

7. **Exterior points.** A point z_0 is exterior to S if it is neither an interior point of S nor a boundary point of S.

8. **Open sets.** A set S is open if all its points are interior points.

   Open sets do not include any of their boundary points.

9. **Closed sets.** A set S is closed if every limit point of S belongs to S.

   This means that a closed set must include its boundary points.

10. **Closure of a set.** The union of a set S with all its limit points is called the closure of S. The closure of S is denoted $\overline{S}$.

    This set is necessarily closed. The notation for the closure uses an overline, like the notation for conjugation, but there should be no confusion since the former is applied to sets and the latter to complex numbers.

11. **Connected sets.** This means exactly what it sounds like; the set has to be all one piece. The precise definition is as follows.

    A set S is connected if, for every two points z_1 and z_2 in S, there exists a path consisting of straight line segments entirely contained in S that connects z_1 to z_2.

    For example, the set $S = \{z \in \mathbb{C} \,:\, |\mathrm{Re}(z)| > 2\}$ is not connected because the points $z_1 = 3$ and $z_2 = -3$ are both in the set, but there is no path connecting them that does not cross the imaginary axis, which is clearly outside S. In contrast, the set $S = \{z \in \mathbb{C} \,:\, |z| > 2\}$ is connected, because, even though points within a distance 2 of the origin are excluded, any two points exterior to the circle radius 2 centered at 0 can be connected by a polygonal path staying outside this circle.

12. **Bounded sets.** A set S is said to be bounded if one can draw a large circle that encloses S. Specifically, if there exists a number $M > 0$ such that $|z| < M$ for all $z \in S$, then S is a bounded set.

13. **Domain/Open region.** A domain or open region is an open connected set.

    This use of the word domain is somewhat specific to the subject of complex analysis. The usual use of the word "domain" is simply to specify the set on which a function is defined, no matter if it is open, closed or not connected. However, in complex analysis when we use the word domain we mean an open connected set.

14. **Region.** A domain with perhaps some or all of its limit points.

To close this section we state two important theorems for bounded sets, whose results we will use several times.

Theorem 2.1 Weierstrass–Bolzano[1] Theorem. *Every bounded infinite set has at least one limit point.*

This theorem basically says, that if you have an infinite sequence of points inside a bounded set, then these points cannot all be greater than some finite distance away from all the others.

Theorem 2.2 *If $S \subset \mathbb{R}^n$ is a closed, bounded set and if g is a continuous function from S to $\mathbb{R}$, then g attains both its maximum and its minimum in S.*

This theorem means there is at least one point $x_1 \in S$ and at least one point $x_2 \in S$ such that $g(x_1) \leq g(x) \leq g(x_2)$, $\forall x \in S$. The theorem does not say that discontinuous functions, or continuous functions on open or unbounded sets *cannot* attain their maximum and minimum values on these sets, but it does *guarantee* that continuous functions on closed and bounded sets do attain their maximum and minimum. As an example, the function $g(x) = x^2 + 1$ attains its minimum value of 1 on $S = [0, 1]$ at $x_1 = 0$ and attains is maximum value of 2 at $x_2 = 1$. However, $g(x) = x^2 + 1$ does not *attain* a maximum on the set $S = [0, 1)$, nor does it on $S = [0, \infty)$ (although it happens to attain its minimum value on these sets). The first set is not closed, the second is not bounded. Also, the discontinuous function defined by $g(x) = 1/x$ if $x \neq 0$ and $g(0) = 0$ does not attain either a maximum or minimum on $S = [-1, 1]$.

Since the complex plane can by put in one-to-one correspondence with $\mathbb{R}^2$, the typical way that Theorem 2.2 will be applied in this text is for a function g that is the modulus of a complex function $f(z)$, that is, $g(x, y) = |f(z)|$, where $z = x + iy$.

2.2 SINGLE-VALUED AND MULTI-VALUED FUNCTIONS

For a set $S \subseteq \mathbb{C}$, a function $f : S \mapsto \mathbb{C}$ is simply a rule that assigns one or more values $w \in \mathbb{C}$ to each input value $z \in S$, that is, $w = f(z)$. The set S is the domain of the function (this use of the word "domain" is the usual one, being the set, not necessarily open and connected, on which the function is defined). If R is any region inside S, then the image of R under f is the set obtained by applying f to each of the points of R. The image of R under f is written

$$f(R) = \{w = f(z) \ : \ z \in R\}.$$

The range of f is the image of the whole domain S. A function f is often referred to as a **mapping** or a **transformation** since it "maps" or "transforms" points in S to points in $f(S)$. As

[1] This theorem is named after Karl Weierstrass (1815–1897) and Bernard Bolzano (1781–1848).

stated earlier, we reserve the symbols x and y for the real and imaginary parts of z and, r and θ for the modulus and argument of z. We therefore need other symbols to represent the parts of w.

Notation convention:
For a complex number w we will often write its Cartesian and polar forms as $w = u + iv$ and $w = \rho e^{i\phi}$. Occasionally, we will also use the Greek letters zeta, xi, and psi (ζ, ξ, and ψ) to represent a complex number, a real number, and an argument, respectively.

In the first sentence of this section you may have noticed and wondered about the phrase "one or more values." You are likely familiar with the idea that functions should provide exactly one output value for each input value. Indeed, this is usually a requirement in the definition of functions for real variables (the "vertical line test"). Such functions are called **single-valued**. For example, the function $f(z) = z^2$ is single-valued since for any input $z = re^{i\theta}$ there is exactly one output $w = r^2 e^{i2\theta}$. In complex analysis we also consider **multivalued** functions. These are functions that return two or more output values for each input value. We have already seen such a function in Section 1.5. For example, the function $f(z) = z^{1/2}$.

$$w = f(z) = z^{1/2} = \left(re^{i\theta}\right)^{1/2} = \left(re^{i(\theta+2\pi k)}\right)^{1/2} = \sqrt{r}\, e^{i(\theta+2\pi k)/2}, \qquad k = 0, 1. \tag{2.1}$$

Thus, for each input value to $f(z) = z^{1/2}$ there are two output values, which differ by a factor $e^{i\pi} = -1$.

A multivalued function can be considered as a collection of single-valued functions; each member is called **branch**. So in the above example, if θ is restricted to lie in a specific interval of length 2π, the function $f(z) = z^{1/2}$ has two branches, one defined by taking $k = 0$ in (2.1) and the other defined by taking $k = 1$ in (2.1). This notion of multiple branches for a function should not be completely new since it is also present for some real-valued functions. For example, the square root function for positive real numbers has two branches: $f_0(x) = \sqrt{x}$ and $f_1(x) = -\sqrt{x}$; and the inverse tangent function has infinitely many branches as we have already discussed in Section 1.1.1. Of the branches making up a multivalued function, generally one branch is denoted as the **principal branch**. For the rational power functions, the principal branch is defined by using the principal argument of the input value. The principal branch of $f(z) = z^{p/q}$ is

$$f(z) = |z|^{p/q}\, e^{i\,\mathrm{Arg}(z)\,p/q}.$$

In the case where z is a positive real number ($\mathrm{Arg}(z) = 0$), then, with this definition, the principal branch is the one that takes the positive qth root of z^p.

A branch of a multivalued function has a line or curve of points where the function is discontinuous. This curve is called a **branch cut**. For example, the principal branch of $z^{p/q}$ is discontinuous along the negative real axis because $\mathrm{Arg}(z)$ has a discontinuity jumping from $-\pi$

to π as the imaginary part of z increases across this curve. In this case the value of $f(z)$ will change by a factor of $e^{i2\pi p/q}$ as the negative real axis is crossed upwards. Branch cuts are not always in the same place. If instead of taking the principal argument of z, for the function $z^{p/q}$, we had selected $\theta \in [\theta_0, \theta_0 + 2\pi)$ for some fixed θ_0, then the branch cut would be located along the ray $\theta = \theta_0$. We will discuss branches and branch cuts more thoroughly in Section 3.5.2.

For many functions $w = f(z)$ we may define an inverse function $z = f^{-1}(w)$. One or both of f and f^{-1} may be multivalued. For example, the function $f(z) = z^2$ is single-valued but is a 2-to-1 function (there are two different inputs, $re^{i\theta}$ and $re^{i(\theta+\pi)}$, that both map to the same output). Hence, it is not surprising that the inverse function $f^{-1}(w) = w^{1/2}$ is a multivalued 1-to-2 function. As another example the function $f(z) = (z^2 + 1)^{1/2}$ is multivalued, as is its inverse $f^{-1}(w) = (w^2 - 1)^{1/2}$.

2.2.1 EXERCISES

2.1. Consider the three branches of the cube root function defined by

$$f_k(z) = |z|^{1/3} e^{i(\operatorname{Arg}(z)+2k\pi)/3}, \qquad -\pi < \operatorname{Arg}(z) \le \pi, \quad k = 0, 1, 2.$$

(a) Find the value of $f_k(-27)$ for each branch, $k = 0, 1, 2$.

(b) Find the value of $f_k(-8i)$ for each branch, $k = 0, 1, 2$.

(c) If the cube root of z is $4e^{-i7\pi/9}$, what is z and what branch (value of k) is being used?

2.2. Consider the principal branch of the square root function, $f(z) = |z|^{1/2} e^{i \operatorname{Arg}(z)/2}$, where $-\pi < \operatorname{Arg}(z) \le \pi$, and a second branch of the square root function given by $g(z) = |z|^{1/2} e^{i(\operatorname{Arg}(z)+2\pi)/2}$.

(a) Show that as z crosses the negative real axis, there is a discontinuous jump in the value of $f(z)$.

(b) Show that as z crosses the negative real axis, the values of $f(z)$ continuously change to the values of $g(z)$.

2.3 LINES AND CIRCLES

Lines and circles are basic shapes that often form the boundaries of regions of interest in $\mathbb{C}$ and so it is worthwhile investigating them.

Equations for lines and circles may be given in several different forms including parametric forms, Cartesian forms, and forms using complex conjugate coordinates, which we now define. Since $z = x + iy$ and $\overline{z} = x - iy$, we may take these two equations and solve for x and y to obtain

$$x = \frac{z + \overline{z}}{2}, \qquad y = \frac{z - \overline{z}}{2i}.$$

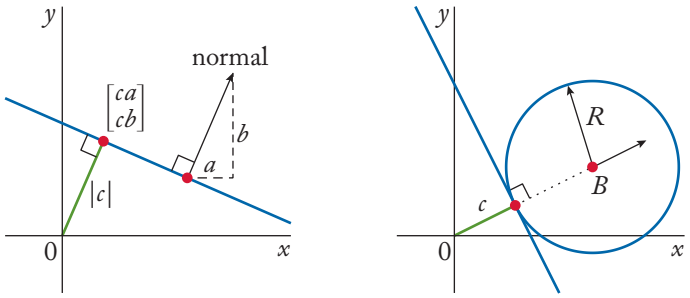


Figure 2.1: Left: The line $ax + by = c$ with normal $[a, b]$ and distance $|c|$ from the origin. Right: The circle $|z - B| = R$ and the line it limits to as $|B|$ and R tend to infinity at the same rate.

The pair $(z, \overline{z})$ are called complex conjugate coordinates and the above transformations indicate how they are related to x and y.

The parametric equation for a line in $\mathbb{C}$ is $z(t) = z_0 + Dt$, where z_0 and D are complex constants, $D \neq 0$, and t is a real parameter. The direction of the line is D, and the line passes through the point z_0 when $t = 0$. If we write $z = x + iy$, then the equation for a line in Cartesian form is $ax + by = c$, where a, b, and c are real constants with the restriction that a and b are not both zero. We can assume that $\sqrt{a^2 + b^2} = 1$ otherwise the equation can be divided by this quantity so that this condition is true. Thinking of $\mathbb{C}$ as $\mathbb{R}^2$, the direction of the line is $\begin{bmatrix} -b \\ a \end{bmatrix}$ and the normal to the line is $\begin{bmatrix} a \\ b \end{bmatrix}$. The point $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ca \\ cb \end{bmatrix}$ is on the line and, due to the conditions on a and b, is a distance $|c|$ from the origin, which is the minimum distance from any point on the line to the origin. See Fig. 2.1. The line passes through 0 if and only if $c = 0$. Changing to complex conjugate coordinates we have

$$\begin{aligned} a\left(\frac{z + \overline{z}}{2}\right) + b\left(\frac{z - \overline{z}}{2i}\right) &= c, \\ a\,(z + \overline{z}) + b(-i)\,(z - \overline{z}) &= 2c, \\ (a - ib)z + (a + ib)\overline{z} &= 2c. \end{aligned}$$

Let $A = a + ib$ be the normal to the line; the assumptions about a and b imply that $|A| = 1$. Thus, the general equation in complex conjugate coordinates for a line on $\mathbb{C}$ with normal A is

$$\overline{A}z + A\overline{z} = 2c, \qquad \text{where } c \in \mathbb{R},\ A \in \mathbb{C},\ |A| = 1, \tag{2.2}$$

and $|c|$ is the minimum distance from the origin to the line. The line will go through 0 if and only if $c = 0$.

Now let us consider circles in the complex plane. The parametric equation for a circle in $\mathbb{C}$ centered at B with radius R is $z(t) = B + Re^{it}$, where B is a complex constant, R is a real

constant, and t is a real parameter that traces out the circle once for each interval of 2π in length. For the Cartesian form of the circle, recall that $|z - B|$ is the distance between z and B. Thus, circles in $\mathbb{C}$ are given by the equation $|z - B| = R$, where $R \in \mathbb{R}$ is the radius and $B \in \mathbb{C}$ is the center of the circle. The circle will pass through zero if $|B| = R$. Now, it is also true that $|w|^2 = w\overline{w}$, therefore we may write the equation of the circle as

$$(z - B)(\overline{z} - \overline{B}) = R^2,$$

which re-arranges to

$$-z\overline{z} + \overline{B}z + B\overline{z} = g, \qquad \text{where } g = |B|^2 - R^2 \in \mathbb{R}. \tag{2.3}$$

The circle passes through 0 if and only if $g = 0$. If you compare (2.2) and (2.3) you will see that lines and circles have very similar forms in complex conjugate coordinates, the differences being the presence of the $z\overline{z}$ term in the circle equation, and the fact that B need not have modulus one.

We now make the claim that lines are simply circles with infinite radius and centered at infinity. This can be seen by taking the circle Eq. (2.3) and dividing by $|B|$. This gives

$$-\frac{z\overline{z}}{|B|} + \frac{\overline{B}}{|B|}z + \frac{B}{|B|}\overline{z} = \frac{|B|^2 - R^2}{|B|} = 0. \tag{2.4}$$

Note that $B/|B| = e^{i\,\mathrm{Arg}(B)}$ and $\overline{B}/|B| = e^{-i\,\mathrm{Arg}(B)}$. Now define the real number $c = |B| - R$, then $|c|$ is the distance from the origin to the nearest point on the circle. Therefore,

$$\frac{|B|^2 - R^2}{|B|} = \frac{(|B| - R)(|B| + R)}{|B|} = c\left(1 + \frac{R}{|B|}\right).$$

We now replace B with $(B + tB/|B|)$ and R with $(R + t)$ for $t \in \mathbb{R}$ and take the limit of (2.4) as t tends to infinity. Then c is a constant and the term in parentheses above approaches 2. Thus, (2.4) becomes

$$0 + e^{-i\,\mathrm{Arg}(B)}z + e^{i\,\mathrm{Arg}(B)}\overline{z} = 2c.$$

Setting $A = e^{i\,\mathrm{Arg}(B)}$ we see that the above is precisely (2.2), hence it is a line. Figure 2.1 illustrates such a circle and its limiting line.

2.3.1 ELEMENTARY MAPPINGS OF LINES AND CIRCLES

In order to help visualize how a function maps a particular region, it is helpful to consider how lines and circles are mapped by elementary operations.

It should be immediately clear that the function $w = f(z) = z + a$ simply shifts any set, including lines and circles, by an amount a in the complex plane. Since this translation does not affect the geometry, lines are mapped to lines in the same direction and circles are mapped to

circles with the same radius. Similarly, the function $w = f(z) = az$, which scales all points by $|a|$ and rotates them around the origin by $\mathrm{Arg}(a)$, maps lines to lines and circles to circles. This time, however, the direction of the line is rotated by $\mathrm{Arg}(a)$, and the radius of the circle is scaled by $|a|$. The center of the circle is also scaled and rotated.

Finally, consider what inversion, $w = f(z) = 1/z$ does to lines and circles. A line with normal A is the set of points satisfying

$$\overline{A}z + A\overline{z} = 2c, \qquad \text{where } c \in \mathbb{R},$$

and a circle with center B and radius R is the set of points satisfying

$$z\overline{z} - \overline{B}z - B\overline{z} = g, \qquad \text{where } g = |B|^2 - R^2 \in \mathbb{R}.$$

The line passes through 0 if $c = 0$ and the circle passes through 0 if $g = 0$. Applying the mapping $w = 1/z$, or equivalently $z = 1/w$ to these we get

$$\overline{A}\frac{1}{w} + A\frac{1}{\overline{w}} = 2c \qquad \Longrightarrow \qquad 2cw\overline{w} - Aw - \overline{A}\overline{w} = 0,$$

and

$$\frac{1}{w\overline{w}} - \overline{B}\frac{1}{w} - B\frac{1}{\overline{w}} = g \qquad \Longrightarrow \qquad gw\overline{w} + Bw + \overline{B}\overline{w} = 1.$$

In the case of the line, we see that if $c = 0$ then the result of the transformation is the equation of a line that passes through 0, and if $c \neq 0$ then the result of the transformation is the equation of a circle with $g = 0$, that is, a circle that passes through 0. In the case of the circle, if $g = 0$ then the transformation results in a line that does not pass through 0, and if $g \neq 0$ then the result is another circle that does not pass through 0. The table below summarizes these results.

set S	image of S under the function f(z) = 1/z
line that passes through 0	line that passes through 0
line that does not pass through 0	circle that passes through 0
circle that passes through 0	line that does not pass through 0
circle that does not pass through 0	circle that does not pass through 0

Considering that the inversion operation maps the modulus from $|z|$ to $1/|z|$, it is intuitively clear that the inversion of a circle that does not pass through 0 must remain bounded, while the inversion of a circle that passes through 0 must be unbounded. Similarly, a line that does not pass through 0 must transform to something bounded while if it passes through 0 then it must be unbounded. Further, since the line itself tends to infinity, its inversion must pass through 0.

Using the interpretation that lines are just circles with infinite radius centered at infinity, we can say that inversion maps all circles to circles.

2.3.2 EXERCISES

2.3. Sketch the following lines and circles and their images under the given mapping f.

(a) $z(t) = 1 + it, t \in \mathbb{R}, \quad f(z) = iz.$

(b) $iz - i\overline{z} = 2, \quad f(z) = \left(-2\sqrt{3} + 2i\right)z - 2 - i.$

(c) $z(t) = -2 - 2i + 4e^{it}, 0 \le t < 2\pi, \quad f(z) = -z - 2.$

2.4. What is the center and radius of the circle obtained by mapping the line $(3 + 4i)z + (3 - 4i)\overline{z} = 4$ by the inversion $f(z) = 1/z$? [Hint: Put the line formula in standard form first by scaling so that $|A| = 1$.]

2.5. What is the distance to the origin and the normal to the line obtained by mapping the circle $z\overline{z} - (2 + 3i)z - (2 - 3i)\overline{z} = 0$ by the inversion $f(z) = 1/z$?

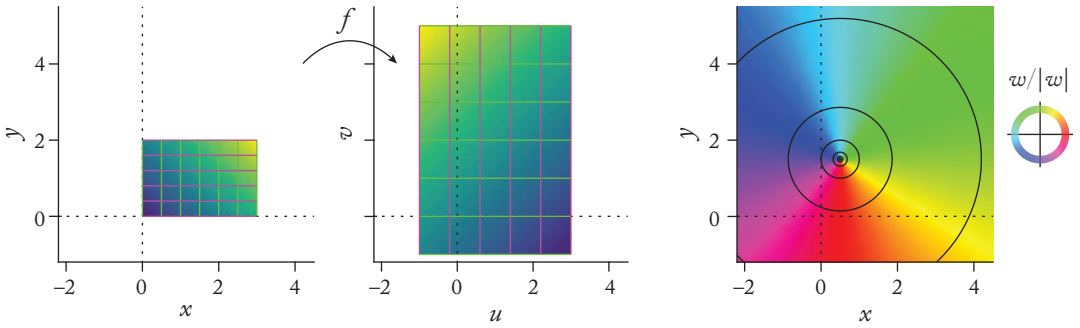
2.6. Show that the effect of inversion by $f(z) = 1/z$ on stereographic projection is a reflection across the $y = 0$ plane and across the $z = 0$ plane for points on the Riemann sphere. That is, if z is stereographically projected to $A = (X, Y, Z)$ on the Riemann sphere, then $w = 1/z$ is stereographically projected to $B = (X, -Y, -Z)$. [Hint: Use the result of Exercise 1.23.]

2.7. Recall from Section 1.6 that we had argued that all lines on $\mathbb{C}$ and circles centered at the origin are stereographically projected to circles on the Riemann sphere. Use the result of the previous exercise to argue that circles in $\mathbb{C}$ that pass through the origin must also stereographically project to circles on the Riemann sphere. (It is also true that circles not passing through the origin get stereographically projected to circles on the Riemann sphere, but that is harder to show.)

2.4 VISUALIZING COMPLEX FUNCTIONS

Let us restrict ourselves to functions f that are single-valued or are a branch of a multivalued function. Since f maps a region of $\mathbb{C}$, which requires two real dimensions to draw, and its output is also in $\mathbb{C}$, requiring another two dimensions to draw, it is more difficult to visualize the action of a complex function than it is to visualize the action of a real function of a single real variable, which can be done by a graph in two dimensions.

One useful way to visualize complex functions is to draw two copies of $\mathbb{C}$, one for the region R and one for the image of R under f. Color and curves can be used to help visualize which points in R correspond to which points in $f(R)$. Consider the function $f(z) = 2iz + 3 - i$ and the region $R = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 3,\ 0 < \mathrm{Im}(z) < 2\}$. Here points in R are first multiplied by $2i$, which doubles the modulus and rotates by $\pi/2$, and then the constant $3 - i$ is added, which shifts the point 3 units to the right and one unit down. The action of this function on R is illustrated in Fig. 2.2a. The color of a point w in the image $f(R)$ is the color of the



(a) The region R (left) and its image under f (right)

(b) Phaseplot of f

Figure 2.2: Illustrations of the function $w = u + iv = f(z) = 2iz + 3 - i$.

preimage $z \in R$ that mapped to w, that is, $f(z) = w$. Thus, for example, the yellow points in the upper-left corner of $f(R)$ are the images of the yellow points in the upper-right corner of R. The horizontal magenta lines in R are mapped to the corresponding magenta lines in $f(R)$, in this case now vertical due to the rotation property of f. Similarly, the green vertical lines in R map to green horizontal lines in $f(R)$.

Another manner of visualizing the action of a complex function is to use a **phase plot**. This method uses a single copy of $\mathbb{C}$. At each point $z \in \mathbb{C}$, color is used to designate the phase of $w = f(z)$, defined as $w/|w| = e^{i \arg w}$. The phase is a point on the unit circle, and colors are chosen to smoothly flow around this circle as illustrated on the right legend in Fig. 2.2b. Contour lines are drawn to show the location of points with constant value of $\log |w|$. Thus, if the contour lines are approximately evenly spaced, this indicates that the modulus is behaving in an exponential manner. Figure 2.2b shows a phase plot for the same example $f(z) = 2iz + 3 - i$. The contour lines are centered at $z_0 = (1 + 3i)/2$, since that is the point that f maps to zero. The contours are concentric circles because the function is linear; all points an equal distance from z_0 are mapped by f to points with the same modulus, which is seen by

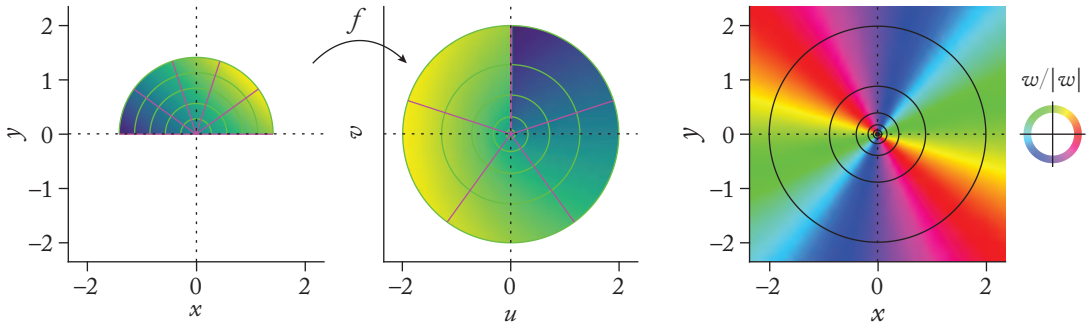
$$|f(z_0 + z)| = |2i(z_0 + z) + 3 - i| = |f(z_0) + 2iz| = |0 + 2iz| = 2|z|.$$

The colors in the plot are centered at z_0, but are seen to be rotated clockwise by $\pi/2$. For example, according to the legend, red corresponds to a zero argument, but in the plot the red points are downward from z_0. This is because the function f maps the vertical line through z_0 to the real axis.

We shall use these techniques for visualizing complex functions in the next section.

2.5 SOME ELEMENTARY FUNCTIONS

Here we list some elementary functions and discuss their properties.



(a) The region R (left) and its image under f (right)

(b) Phaseplot of f

Figure 2.3: Illustrations of the function $w = u + iv = f(z) = iz^2$.

2.5.1 POLYNOMIALS

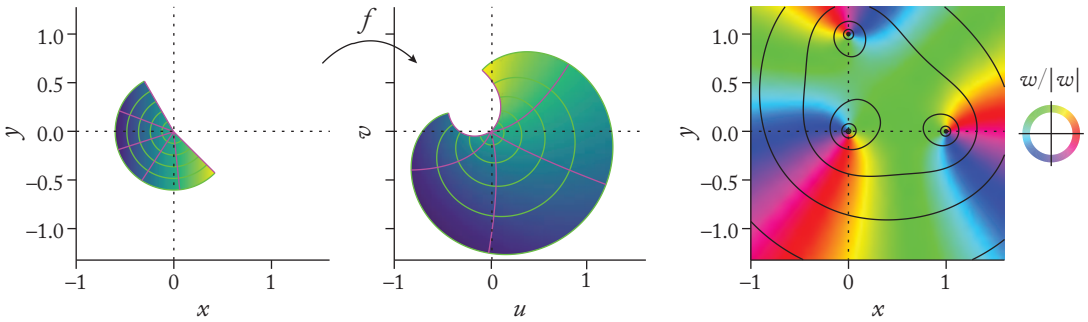
Polynomials are functions of the form

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = \sum_{k=0}^{n} a_k z^k,$$

where $a_k \in \mathbb{C}$, $0 \le k \le n$, $a_n \neq 0$. The degree of the polynomial, denoted $\deg f$, is n; if $n = 1$ we say the polynomial is *linear*. (Although the polynomial is called "linear," it does not define a linear map unless $a_0 = 0$; technically it is called an *affine* mapping.) A linear polynomial scales each point z by $|a_1|$, rotates by $\mathrm{Arg}(a_1)$, and then translates by a_0. As noted above, it maps lines to lines and circles to circles. An example of a linear polynomial was given in the previous section. The next example considers a simple quadratic.

Example 2.3 Consider the function $f(z) = iz^2$ and the region $R = \left\{ z \in \mathbb{C} \,:\, 0 \le \mathrm{Arg}(z) \le \pi,\ |z| < \sqrt{2} \right\}$. The action of squaring z squares its modulus and doubles its angle, thus points in R will result in modulus values between 0 and 2, and arguments in the range $[0, 2\pi]$. Then this result is multiplied by i, which has the effect of rotating the points by $\pi/2$ counter-clockwise. This action is illustrated in Fig. 2.3. As one circles the origin in the phase plot of this figure, two copies of all the phase circle colors are encountered, due to the fact that the range of arguments $[0, 2\pi)$ will double to $[0, 4\pi)$. What do you think a phase plot of $f(z) = z^3$ would look like?

Although the simple example of $f(z) = iz^2$ maps circles centered at the origin to circles centered at the origin, it does not in general map other circles to circles. For example, the circle of radius 1, centered at 1 includes the points 2, $1 + i$, 0, and $1 - i$, but these four points map to $4i$, -2, 0, and 2, respectively. Clearly, these four points do not lie on a circle.



(a) The region R (left) and its image under f (right)

(b) Phaseplot of f

Figure 2.4: Illustrations of the function $w = u + iv = f(z) = z(z - i)(z - 1)$.

The next example illustrates a cubic polynomial.

Example 2.4 Consider the function $f(z) = z(z - i)(z - 1)$ and the region $R = \{z \in \mathbb{C} : 2\pi/3 \leq \text{Arg}(z) \leq 7\pi/4,\ |z| \leq 0.6\}$. This function has three zeros, at 0, i, and 1, which are immediately evident in the phase plot; Fig. 2.4b. Each of the zeros is not repeated, so, unlike in Example 2.3, as one circles one of the zeros, the colors appear only once. In this example it is more difficult to obtain information from the image plot; Fig. 2.4a. However, it is clear from that plot as well as the phase plot, that points left and downward from the origin get spread out in the angular direction and are rotated about $\pi/2$ counter-clockwise. This is because the factor $(z - i)(z - 1)$ is approximately i for z close to zero.

2.5.2 RATIONAL FUNCTIONS

Rational Functions are functions of the form $f(z) = \dfrac{P(z)}{Q(z)}$, where P and Q are polynomials. In general, the degrees of P and Q can be anything, but the case where both are degree at most one is an important and well-studied situation.

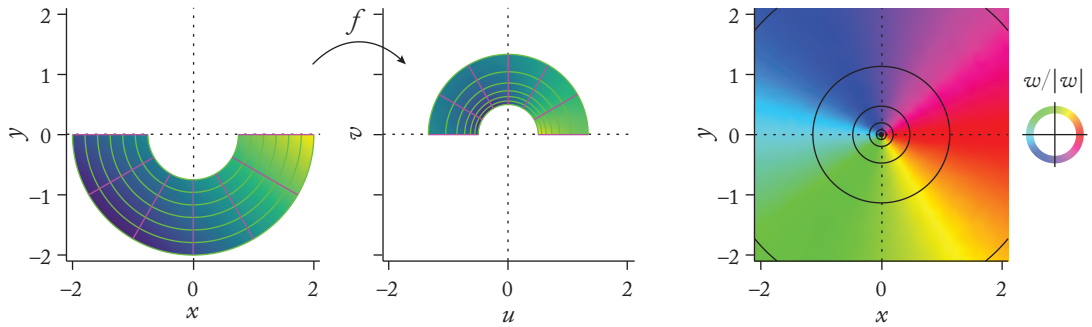
If P and Q are linear polynomials or constants, the rational function is called a **linear fractional transformation**, or **Möbius transformation**.[2] Such a transformation can be written as

$$w = f(z) = \frac{az + b}{cz + d}. \tag{2.5}$$

Re-arranging this equation gives

$$czw + dw - az - b = 0. \tag{2.6}$$

[2] August Ferdinand Möbius (1790–1868) was a German mathematician working mostly in theoretical astronomy.



(a) The region R (left) and its image under f (right)

(b) Phaseplot of f

Figure 2.5: Illustrations of the function $w = u + iv = f(z) = 1/z$.

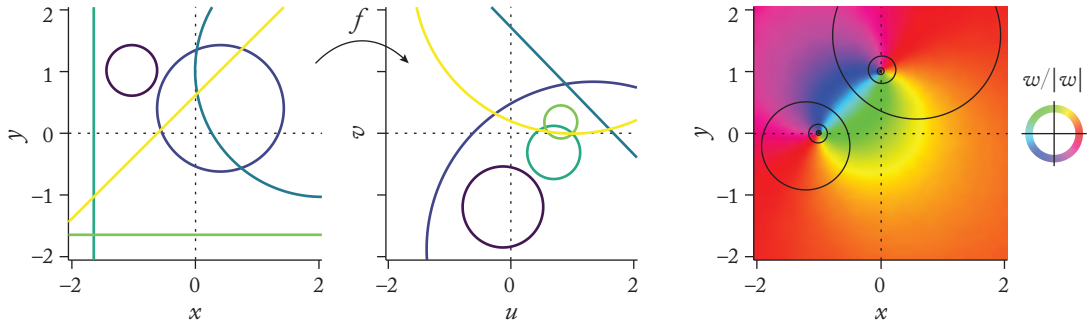
Since this last equation is linear in z and linear in w, this transformation is also called a **bilinear transformation**. By direct computation of df/dz, it is easy to show that f is constant if $ad - bc = 0$. Hence one generally assumes that $ad - bc \neq 0$ for these transformations. A linear polynomial transformation (discussed in the previous section) is the special case $c = 0$, and inversion (times a constant) is the special case $a = d = 0$.

Example 2.5 Consider $f(z) = \frac{1}{z}$ and the region $R = \{z \in \mathbb{C} : -\pi < \text{Arg}(z) \leq 0,\ 3/4 \leq |z| \leq 2\}$. Since $f(re^{i\theta}) = \frac{1}{r}e^{-i\theta}$, we see that this function will invert the modulus and negate the argument. Thus, R maps into the region bounded by the circles with radius $1/2$ and $4/3$ and with arguments between 0 and π, as illustrated in Fig. 2.5. As one circles the origin in the phase plot of this figure, the color ordering is reversed due to the negation of the argument.

Now consider the general linear fractional transformation (2.5). Setting $\zeta = cz + d$, which implies $z = \dfrac{\zeta - d}{c}$, we may write a linear fractional transformation as

$$w = f(z) = \frac{a\left(\frac{\zeta - d}{c}\right) + b}{\zeta} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{\zeta}.$$

This function is a linear polynomial in the variable $1/\zeta$. Therefore, the action of a linear fractional transformation is first to apply a linear polynomial (scale, rotate, and translate) producing ζ, then invert producing $1/\zeta$, then apply another linear polynomial to $1/\zeta$ (scale, rotate, and translate) producing the final result. In Section 2.3.1 we showed that circles (including lines, which are circles with infinite radius) are mapped to circles by both linear polynomials and by inversion. Therefore, since linear fractional transformations are a composition of these two kinds of mappings, we conclude that linear fractional transformations also map circles to circles.



(a) Various lines and circles (left) and their images under f (right)

(b) Phaseplot of f

Figure 2.6: Illustrations of the function $w = u + iv = f(z) = (z + 1)/(z - i)$.

Example 2.6 Consider the linear fractional transformation $f(z) = \dfrac{z+1}{z-i}$. Various circles and lines, and their images under f, are shown in Fig. 2.6a. In every case the image is a circle or line. All three of the lines in the preimage map to circles that pass through the point $w = 1$, because as $|z|$ gets large, the value of $f(z)$ approaches 1. Figure 2.6b shows the phase plot for this function. The colors are in their positive ordering around the point $z = -1$, which is a zero of the numerator, but are in the negative ordering around $z = i$, which is a zero of the denominator. As one moves far away from the origin, the value of $f(z)$ approaches 1, and hence the coloring is approaching red.

We now claim that it is possible to find a linear fractional transformation that maps any given circle to any other given circle. Again here we consider lines as circles with infinite radius. From geometry we know that any three points on a circle uniquely defines it (the center is the intersection of the perpendicular bisectors of two different chords each joining two of the three points). If the three points are colinear then the two bisectors are parallel and the center is the point at infinity. Thus, our claim is equivalent to saying that there exists a linear fractional transformation that maps any three distinct points z_1, z_2, and z_3, to any three distinct points w_1, w_2, and w_3. This can be achieved by the transformation

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}. \tag{2.7}$$

Multiplying both sides of the above by the four factors in the denominator yields

$$(z - z_3)(z_2 - z_1)(w - w_1)(w_2 - w_3) = (z - z_1)(z_2 - z_3)(w - w_3)(w_2 - w_1). \tag{2.8}$$

It is now immediately clear that both sides of the above are products of linear polynomials in z and w and hence the equation can be simplified to the bilinear form (2.6); thus it is a linear fractional transformation. Since the z_j are distinct sets of points, none of the differences of these points is zero, and similarly for the w_j. Consequently, from (2.8) we see that when $z = z_1$ the right side is zero and so necessarily $w = w_1$. Similarly, when $z = z_3$, the left side of (2.8) is zero implying $w = w_3$. Finally, when $z = z_2$ Eq. (2.8) collapses to $(w - w_1)(w_2 - w_3) = (w - w_3)(w_2 - w_1)$, whose unique solution is $w = w_2$. Therefore, the transformation (2.7) is a linear fractional transformation that maps z_j to w_j, $1 \le j \le 3$. Since such transformations map circles to circles, the circle through the points z_j necessarily maps to the circle through the points w_j.

Example 2.7 Find the linear fractional transformation that maps the points $z_1 = 1$, $z_2 = 0$, and $z_3 = i$ to the points $w_1 = 0$, $w_2 = -1$, and $w_3 = -i$, respectively.

Substituting these values into (2.8) and simplifying gives

$$(-1 + 2i)zw + w - z + 1 = 0 \qquad \Longrightarrow \qquad w = \frac{z - 1}{(-1 + 2i)z + 1}.$$

A linear fractional transformation $w = f(z)$ given by (2.5) with $ad - bc \neq 0$ is defined for every complex number except $z = -d/c$ if $c \neq 0$. Further, solving (2.5) for z gives

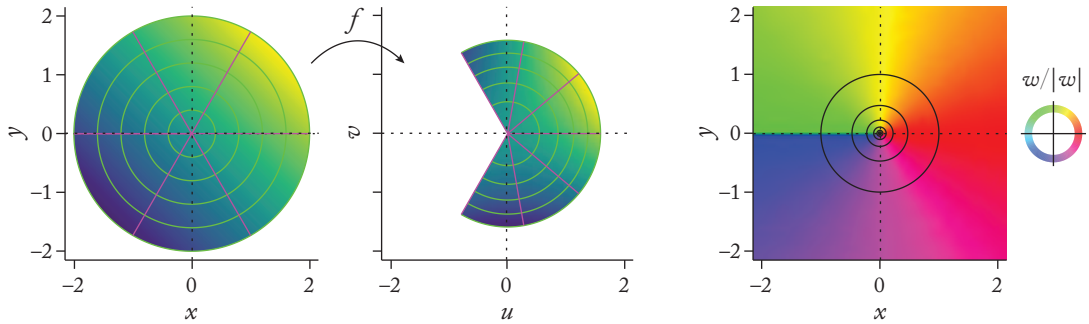
$$z = \frac{-dw + b}{cw - a}. \tag{2.9}$$

Therefore, a preimage z exists for every point $w \in \mathbb{C}$ except the point $w = a/c$ if $c \neq 0$. It is possible to make this transformation one-to-one by expanding its domain and range to the extended complex plane. Assuming $ad - bc \neq 0$, define

$$\text{if } c = 0\text{:} \quad f(z) = \begin{cases} \infty & \text{if } z = \infty, \\ \frac{az+b}{cz+d} & \text{if } z \neq \infty; \end{cases} \qquad \text{if } c \neq 0\text{:} \quad f(z) = \begin{cases} \infty & \text{if } z = -d/c, \\ \frac{a}{c} & \text{if } z = \infty, \\ \frac{az+b}{cz+d} & \text{otherwise.} \end{cases} \tag{2.10}$$

That this extension of f is continuous is left as an exercise. Since this extended function is one-to-one, it is invertible, and from (2.9) we see that the inverse is also a linear fractional transformation:

$$\text{if } c = 0\text{:} \quad f^{-1}(w) = \begin{cases} \infty & \text{if } w = \infty, \\ \frac{-dw+b}{cw-a} & \text{if } w \neq \infty; \end{cases} \qquad \text{if } c \neq 0\text{:} \quad f^{-1}(w) = \begin{cases} \infty & \text{if } w = a/c, \\ \frac{-d}{c} & \text{if } w = \infty, \\ \frac{-dw+b}{cw-a} & \text{otherwise.} \end{cases}$$



(a) The region R (left) and its image under f (right)

(b) Phaseplot of f

Figure 2.7: Illustrations of the function $w = u + iv = f(z) = z^{2/3}$ (principal branch).

2.5.3 RATIONAL POWERS

As discussed in Section 1.5, for a rational number p/q such that p and q share no common factors, the rational function $f(z) = z^{p/q}$ is multivalued with q branches given by

$$f(z) = z^{p/q} = r^{p/q} e^{i(\theta+2\pi k)p/q}, \qquad 0 \le k < q,$$

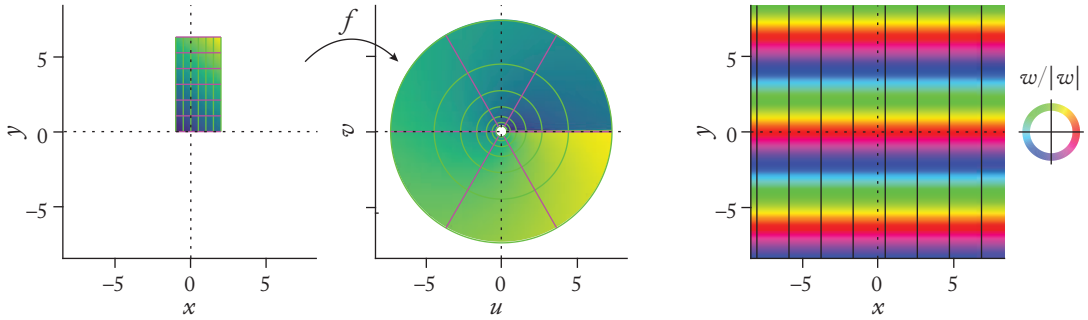
where $z = re^{i\theta}$, and θ is in some interval of length 2π. The principal branch is the one with $\theta = \mathrm{Arg}(z)$ and $k = 0$.

Example 2.8 Consider the principal branch of $f(z) = z^{2/3}$ and the region $R = \{z \in \mathbb{C} : |z| \le 1\}$. This function is illustrated in Fig. 2.7. The image of R only includes arguments from $-2\pi/3$ to $2\pi/3$. This is also visible in the phase plot since the colors jump from green to dark blue across the negative real axis. The negative real axis can thus be seen to be a line of discontinuity for this function.

2.5.4 THE EXPONENTIAL

The exponential function was introduced in Section 1.2. Since $e^z = e^{x+iy} = e^x e^{iy}$, horizontal lines (where y is constant) map to a ray from the origin at angle y. Similarly, vertical lines (where x is constant) map to circles of radius e^x centered at the origin. Further, each 2π interval for y will constitute another traversal of the circle. This is the $2\pi i$ periodic nature of the exponential function.

Example 2.9 Consider $f(z) = e^z$ and the region $R = \{z \in \mathbb{C} : -1 \le \mathrm{Re}(z) \le 2,\ 0 \le \mathrm{Im}\, z < 2\pi\}$. This region and its image under $f(z)$ are shown in Fig. 2.8a. From this it is clearly seen how the horizontal lines map to rays and



(a) The region R (left) and its image under f (right)

(b) Phase plot of f

Figure 2.8: Illustrations of the function $w = u + iv = f(z) = e^z$.

the vertical lines to circles. The phase plot in Fig. 2.8b shows the $2\pi i$ periodic nature of the exponential function. Further, the regularly spaced vertical contour lines indicate that the modulus is increasing exponentially with x, as expected.

2.5.5 TRIGONOMETRIC FUNCTIONS

The trigonometric functions are defined in terms of the exponential:

$$\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2}, & \sin z &= \frac{e^{iz} - e^{-iz}}{2i}, & \tan z &= \frac{\sin z}{\cos z}, \\ \sec z &= \frac{1}{\cos z}, & \csc z &= \frac{1}{\sin z}, & \cot z &= \frac{1}{\tan z}. \end{aligned} \tag{2.11}$$

Note the factor i in the denominator of the definition for sin, and that it is not in the denominator of the definition for cos. Properties of the real trigonometric functions also hold for these complex extensions. For example, the Pythagorean identity still holds:

$$\begin{aligned} \sin^2 z + \cos^2 z &= \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 \\ &= \frac{e^{i2z} - 2 + e^{-i2z}}{-4} + \frac{e^{i2z} + 2 + e^{-i2z}}{4} = 1. \end{aligned}$$

From this it follows that $\tan^2 z + 1 = \sec^2 z$ and $1 + \cot^2 z = \csc^2 z$. It is also true that $\sin(z)$ is an odd function and $\cos(z)$ is an even function, that is

$$\sin(-z) = -\sin(z), \qquad \cos(-z) = \cos(z).$$

From this it follows that $\tan(z)$ and $\csc(z)$ are odd and $\sec(z)$ is even. The usual sum and difference of angles formulas also hold. For example,

$$\begin{aligned}\sin(z_1)\cos(z_2)+\cos(z_1)\sin(z_2) &= \left(\frac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right)+\\ &\quad \left(\frac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2}-e^{-iz_2}}{2i}\right)\\ &= \frac{e^{iz_1}e^{iz_2}-e^{-iz_1}e^{-iz_2}}{2i} = \frac{e^{i(z_1+z_2)}-e^{-i(z_1+z_2)}}{2i}\\ &= \sin(z_1+z_2).\end{aligned}$$

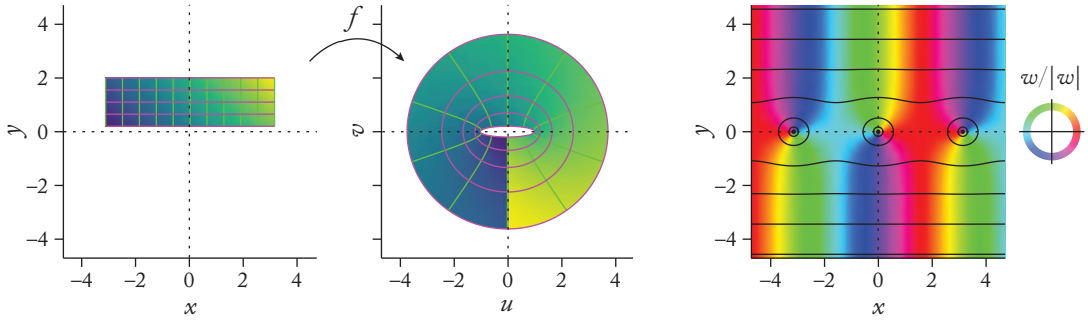
The other sum and difference angle formulas also hold.

Example 2.10 Consider the function $f(z)=\sin z$. Figure 2.9a shows the region $R=\{z\in\mathbb{C} : -\pi\le \mathrm{Re}(z)\le\pi,\ 0.2\le \mathrm{Im}(z)\le 2\}$ and its image under f, while a phase plot of f is shown in Fig. 2.9b. The top boundary of the region R is mapped to the outer, roughly circular, boundary of $f(R)$. The two vertical boundaries of R are both mapped to the segment on the negative imaginary axis in $f(R)$. The bottom boundary of R is mapped to the inner, roughly elliptical, boundary of $f(R)$. If the region R were extended down to the real axis, then this inner boundary of $f(R)$ would collapse to the segment of the real axis between -1 and 1. In the phase plot, Fig. 2.9b, we see that the contour lines are roughly circular and the colors do one full circuit around $z=k\pi$, $k\in\mathbb{Z}$, reflecting the fact that $\sin z$ is approximately $z-k\pi$ near $k\pi$. As one moves away from these points along the real axis, the modulus hits a maximum of 1 and then decreases back to zero by time the next zero is reached. In contrast, as one moves away from $z=k\pi$ in the vertical direction, the modulus continues to increase since either e^{iz} or e^{-iz} will have a large real component when $z=k\pi\pm iy$ and $|y|$ is large. The real axis is always colored either red or light blue, indicating that the argument of $\sin z$ is always 0 or π when z is real. The 2π-periodic nature of $\sin z$ is evident from the colors in the phase plot since a horizontal shift of 2π gives the same image.

2.5.6 HYPERBOLIC FUNCTIONS

The hyperbolic functions are also defined in terms of the exponential function:

$$\begin{aligned}\cosh z &= \frac{e^z+e^{-z}}{2}, & \sinh z &= \frac{e^z-e^{-z}}{2}, & \tanh z &= \frac{\sinh z}{\cosh z},\\ \mathrm{sech}\, z &= \frac{1}{\cosh z}, & \mathrm{csch}\, z &= \frac{1}{\sinh z}, & \coth z &= \frac{1}{\tanh z}.\end{aligned} \tag{2.12}$$



(a) The region R (left) and its image under f (right)

(b) Phaseplot of f

Figure 2.9: Illustrations of the function $w = u + iv = f(z) = \sin z$.

Unlike for sin, there is no i in the denominator of the definition for sinh. Properties of the hyperbolic functions include

$$\cosh^2 z - \sinh^2 z = 1 \quad 1 - \tanh^2 z = \operatorname{sech}^2 z, \quad \sinh(-z) = -\sinh(z), \quad \cosh(-z) = \cosh(z).$$

Finally, the relationship between the hyperbolic functions and the trigonometric functions are:

$$\sin(iz) = i\sinh(z), \qquad \sinh(iz) = i\sin(z), \qquad \cos(iz) = \cosh(z), \qquad \cosh(iz) = \cos(z).$$

Establishing these relationships is left as an exercise.

2.5.7 THE LOGARITHMIC FUNCTION

The logarithmic function is very important. Since it is the inverse function of the exponential, and since the exponential function is $2\pi i$ periodic, we expect the logarithmic function will be multivalued with values differing by $2\pi i$ on each branch. Let $z = re^{i\theta}$ and let $\log z = u + iv$, where u and v are real. As the inverse of the exponential, we require the logarithmic function to satisfy $e^{\log z} = z$. This means

$$re^{i\theta} = z = e^{\log z} = e^{u+iv} = e^u e^{iv}.$$

Therefore, $r = e^u$, that is, $u = \ln r$, ($r > 0$ and u are real), and $v = \theta + 2\pi k$, $k \in \mathbb{Z}$. Thus, the definition of the complex logarithmic function for nonzero z is

$$\begin{aligned} \log z &= \ln r + i(\theta + 2\pi k), && r > 0,\ k \in \mathbb{Z}, \\ &= \ln|z| + i\arg(z), && z \neq 0. \end{aligned}$$

Here we use the notation $\ln r$ for the real logarithmic function, defined as the inverse of the real exponential function. As in the real case, the notations ln and log can be used interchangeably

for the complex logarithmic function, and most texts do exactly that. In this text, however, to help emphasize the multivalued nature of the complex logarithm, we shall restrict the use of ln to the case where we are talking about the real logarithmic function, and we shall only use the notation log when we are talking about the complex logarithmic function.

Because the values of the logarithmic function differ by multiples of $2\pi i$, and since the exponential function is $2\pi i$-periodic, it follows that $e^{\log z}$ evaluates to a single value, namely z. However, in the other direction we have

$$\begin{aligned} \log(e^z) &= \log(e^{x+iy}) = \log(e^x\, e^{iy}) = \ln e^x + i(y + 2\pi k), \quad k \in \mathbb{Z}, \\ &= x + iy + i2\pi k = z + i2\pi k. \end{aligned} \tag{2.13}$$

This means that if you exponentiate a number z, and then take the logarithm, *one* of the solutions you get back will be z, but you will also get other solutions differing by multiples of $2\pi i$.

A branch of the multivalued logarithmic function can be defined by restricting $\arg(z)$ to lie in any particular interval of length 2π, for example, $0 \le \arg(z) < 2\pi$, or $-\pi/2 < \arg(z) \le 3\pi/2$. The **principal branch** of the logarithmic function, denoted Log, is defined using the principal argument via

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg}(z).$$

Thus, the imaginary part of $\operatorname{Log} z$ will lie in the interval $(-\pi, \pi]$. The **principal value** of $\log z$ is the value obtained from the principal branch.

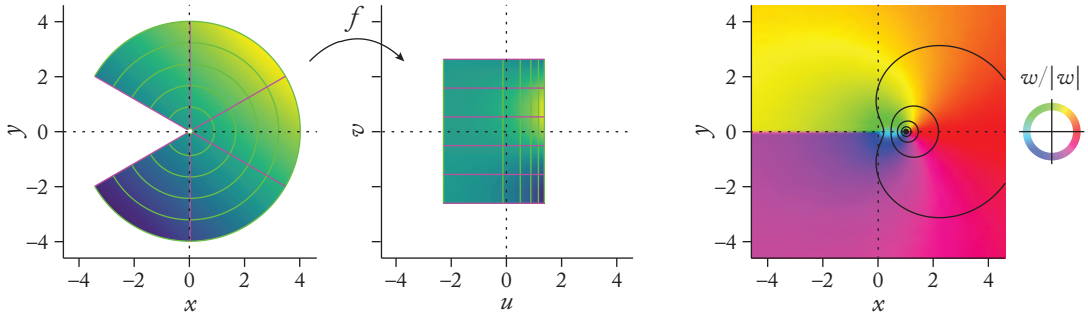
The logarithmic function maps circles of radius r centered at 0 to vertical lines with x-coordinate equal to $\ln r$, and it maps rays from the origin with angle θ from the positive real axis to horizontal lines with y-coordinate equal to $\theta + 2\pi k$.

The complex logarithm satisfies the familiar properties:

$$\log(z_1 \cdot z_2) = \log z_1 + \log z_2, \qquad \log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2.$$

These can be proved from the definition without much difficulty.

Example 2.11 Figure 2.10 illustrates the effect of mapping the region $R = \{z \in \mathbb{C} : 0.1 \le |z| \le 4,\ -5\pi/6 < \operatorname{Arg}(z) \le 5\pi/6\}$ by the function $f(z) = \operatorname{Log} z$. As expected, rays are mapped to horizontal lines and circles to vertical lines, the opposite of the exponential function. The upper boundary of $f(R)$ is the image of the ray $\theta = 5\pi/6$ in R since the imaginary part of $\operatorname{Log} z$ is its principal argument. In the phase plot of Fig. 2.10b, note that the colors on the left half of the phase circle (greens and blues) only appear in the region $|z| < 1$, since the real part of $\operatorname{Log} z$ is $\ln|z|$ and this is only negative when $|z| < 1$. The discontinuity for $\operatorname{Log} z$ across the negative real axis is also clear in this figure by the color jump from yellow to magenta.



(a) The region R (left) and its image under f (right)

(b) Phaseplot of f

Figure 2.10: Illustrations of the function $w = u + iv = f(z) = \operatorname{Log} z$.

2.5.8 COMPLEX POWERS

For $z, \alpha \in \mathbb{C}$, $z \neq 0$, complex powers are defined as

$$z^\alpha = e^{\alpha \log z}.$$

Since log is multivalued, complex powers are also multivalued. The **principal value** is the one associated with the principal branch of the logarithm. The familiar property $\ln x^a = a \ln x$ for the real logarithm is slightly modified in the complex case to

$$\log z^\alpha = \alpha \log z + i2\pi k, \qquad k \in \mathbb{Z}. \tag{2.14}$$

This follows from the definition of z^α and from (2.13) since

$$\log z^\alpha = \log(e^{\alpha \log z}) = \alpha \log z + i2\pi k.$$

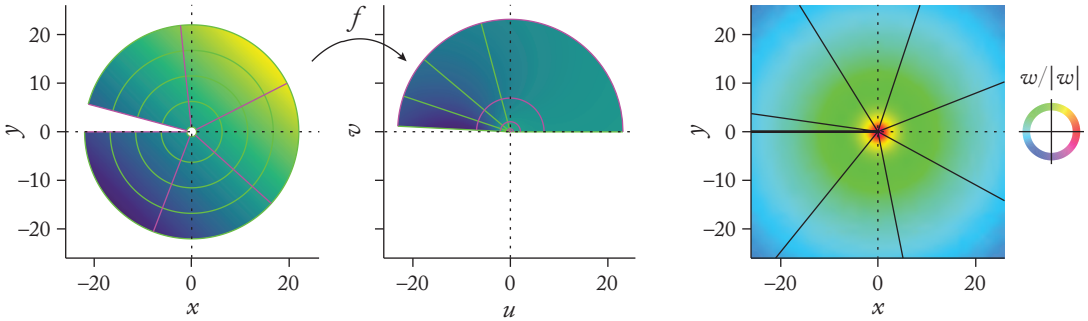
Using this definition of a complex power, we may also obtain formulas for logarithmic functions with bases other than e. For $\alpha \in \mathbb{C}$, $\alpha \neq 0$, we define $w = \log_\alpha z$ by requiring that $\alpha^w = z$. Then

$$z = \alpha^w = e^{w \log \alpha} \quad \Longrightarrow \quad \log z = w \log \alpha, \quad \Longrightarrow \quad w = \frac{\log z}{\log \alpha},$$

that is,

$$\log_\alpha z = \frac{\log z}{\log \alpha}.$$

Example 2.12 Consider the function $f(z) = z^i = e^{i \operatorname{Log} z} = e^{-\operatorname{Arg}(z) + i \ln|z|}$. Figure 2.11a illustrates the effect of this function on the region $R = \{z \in \mathbb{C} \,:\, 1 \leq |z| \leq 22,\ -\pi < \operatorname{Arg}(z) \leq 11\pi/12\}$. If $\operatorname{Arg}(z)$ is constant, then the modulus



(a) The region R (left) and its image under f (right)

(b) Phaseplot of f

Figure 2.11: Illustrations of the function $w = u + iv = f(z) = z^i$.

of $f(z)$ is constant. Therefore, rays map to circles. The ray $\theta = -\pi$ maps to the large circle with radius $e^{\pi} \approx 23$; the ray $\theta = 11\pi/12$ maps to the small circle with radius $e^{-11\pi/12} \approx 0.06$. Similarly, circles map to rays because if $|z|$ is constant then the argument of $f(z)$ is constant. In particular, the circle with radius 1 (inner boundary of R) maps to a ray with argument $\ln 1 = 0$, which is the positive real axis, and the circle of radius 22 maps to a ray with argument $\ln 22 \approx 0.98\pi$. The phase plot in Fig. 2.11b has contour lines that are rays from the origin, again indicating that the modulus of $f(z)$ is constant along rays. Although the color is continuous across the negative real axis, indicating that the argument is continuous, the modulus of $f(z)$ is discontinuous across this ray, since $|f(z)|$ jumps from a value of e^{π} to $e^{-\pi}$ as z moves downward across it. (The thick black line in the plot along the negative real axis is a numerical artifact due to this discontinuity in the modulus; every contour line tries to go along it.)

2.5.9 INVERSE TRIGONOMETRIC FUNCTIONS

The inverse trigonometric functions are all defined in terms of the logarithmic function:

$$\arcsin z = -i \log\left(iz + (1-z^2)^{1/2}\right), \qquad \arccos z = -i \log\left(z + i(1-z^2)^{1/2}\right),$$

$$\arctan z = \frac{i}{2} \log\left(\frac{i+z}{i-z}\right), \qquad \operatorname{arcsec} z = -i \log\left(\frac{1 + i(z^2-1)^{1/2}}{z}\right),$$

$$\operatorname{arccsc} z = -i \log\left(\frac{i + (z^2-1)^{1/2}}{z}\right), \qquad \operatorname{arccot} z = \frac{i}{2} \log\left(\frac{z-i}{z+i}\right).$$

Each of these expressions is derived in a similar manner; we shall do the first:

$$w = \arcsin z \implies z = \sin w \implies 2iz = e^{iw} - e^{-iw} \implies e^{2iw} - 2ize^{iw} - 1 = 0.$$

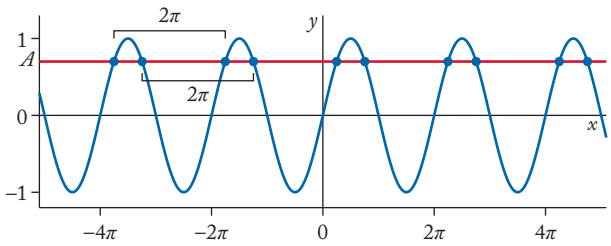


Figure 2.12: The real sine function has two infinite sequences of points that are solutions of $\sin x = A$ for $|A| < 1$. In each sequence, the points differ by multiples of 2π.

This last equation is a quadratic equation for the variable e^{iw}, therefore

$$e^{iw} = iz + \left((iz)^2 + 1\right)^{1/2} \quad \Longrightarrow \quad w = \frac{1}{i}\log\left(iz + (1 - z^2)^{1/2}\right).$$

Note that $\arcsin z$ is multivalued for two reasons: the square root gives two values and then the $\log$ function provides and infinite sequence of values for each of these. Even in the real case this is exactly what one would expect as shown in Fig. 2.12. For a fixed value A, the line $y = A$ intersects $y = \sin x$ in two places in any interval of length 2π; this corresponds to the two values from the square root function. Then, because $\sin x$ is 2π-periodic, every multiple of 2π added to one of these solutions will also be a solution. These are the infinite number of values generated by the logarithmic function.

The inverse trigonometric functions also satisfy

$$\operatorname{arccsc} z = \arcsin\left(\frac{1}{z}\right), \qquad \operatorname{arcsec} z = \arccos\left(\frac{1}{z}\right), \qquad \operatorname{arccot} z = \arctan\left(\frac{1}{z}\right).$$

2.5.10 INVERSE HYPERBOLIC FUNCTIONS

The inverse hyperbolic functions are also defined in terms of the logarithm:

$$\begin{aligned}
\operatorname{arcsinh} z &= \log\left(z + (z^2 + 1)^{1/2}\right), & \operatorname{arccosh} z &= \log\left(z + (z^2 - 1)^{1/2}\right), \\
\operatorname{arctanh} z &= \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), & \operatorname{arcsech} z &= \log\left(\frac{1 + (1 - z^2)^{1/2}}{z}\right), \\
\operatorname{arccsch} z &= \log\left(\frac{1 + (z^2 + 1)^{1/2}}{z}\right), & \operatorname{arccoth} z &= \frac{1}{2}\log\left(\frac{z+1}{z-1}\right).
\end{aligned} \tag{2.15}$$

These expressions are developed similar to how $\arcsin$ was developed above.

2.5.11 EXERCISES

2.8. Sketch the image of the given set under the given mapping $f(z)$.

(a) $|z| \leq 2, 0 \leq \arg z \leq \pi/4$; $f(z) = z^2 - 2i$.

(b) $z = -1 + ti, t \in \mathbb{R}$; $f(z) = z^2 + 3z + 6 - i$.

2.9. Show that the transformation on the extended complex plane defined by (2.10) is continuous.

2.10. Show that all linear fractional transformations that map the upper half-plane $y > 0$ to the open disk $|w| < 0$, and map the boundary $y = 0$ to the boundary $|w| = 1$ can be written in the form

$$w = e^{i\alpha}\frac{z - z_0}{z - \overline{z_0}},$$

where $\alpha \in \mathbb{R}$ and $\operatorname{Im} z_0 > 0$. [Hint: Start by determining the conditions on the transformation that must be imposed so that $z = 0$, $z = 1$ and $z = \infty$ map to points with unit modulus.]

2.11. Using Euler's formula, show that if $z = x + i0$ in the definition for $\sin z$ given by (2.11), then the expression collapses to $\sin x$ as expected.

2.12. Establish the following identities using the definitions of the trigonometric and hyperbolic functions in terms of the exponential function, and $z = x + iy$.

(a) $\cos(iz) = \cosh z$, **(b)** $\sin(iz) = i \sinh z$,

(c) $\cosh^2 z - \sinh^2 z = 1$, **(d)** $\sin z = \sin x \cosh y + i \cos x \sinh y$.

2.13. Show that

(a) $\cos(i\,\overline{z}) = \overline{\cos(iz)}$ for all z;

(b) $\sin(i\,\overline{z}) = \overline{\sin(iz)}$ if and only if $z = n\pi i, n \in \mathbb{Z}$.

2.14. From the definition of the logarithmic function, establish the following.

(a) $\log(z_1 \cdot z_2) = \log z_1 + \log z_2$, **(b)** $\log\left(\dfrac{z_1}{z_2}\right) = \log z_1 - \log z_2$.

2.15. Show that the function $f(z) = \sin z$ maps the semi-infinite strip $R = \left\{z \in \mathbb{C} : |x| \leq \frac{\pi}{2},\ y \geq 0\right\}$ into the upper half-plane. To which points to the boundaries of R map? To which points does the positive imaginary axis map?

2.16. Let R be the half disk, radius 4 centered at the origin and located in the right half-plane. For each of the following functions f, sketch the image of R under the mapping defined by f.

(a) $f(z) = z^2$

(b) principal branch of $f(z) = z^{1/2}$ (branch cut on the negative real axis)

(c) the branch of $f(z) = z^{1/2}$ with $f(1) = 1$ and $0 \le \arg(z) < 2\pi$, (branch cut on the positive real axis)

(d) $f(z) = \operatorname{Log}(z)$ (the principal branch of the logarithm)

2.17. Let log be the branch of the logarithmic function defined by $\log z = \ln|z| + i\arg(z)$ where $0 \le \arg(z) < 2\pi$. Compute the following.

(a) $\log(-1)$, **(b)** $\log(-ei)$, **(c)** $\log(1-i)$.

2.18. Let n be an integer.

(a) Using de Moivre's theorem, prove that

$$\log(z^n) = n\log(z) \tag{2.16}$$

if and only if $n = \pm 1$. That is, the set of values given by the multivalued function on the left is the same as the set of values given by the function on the right only in the cases $n = \pm 1$.

(b) In the case $|n| \neq 1$, how are the two sides of (2.16) related? That is, is one set contained in the other or do both sets contain elements not in the other?

(c) Consider (2.16) but replace log with Log on both sides. Under what situations does this new equation hold?

2.19. Find the principal value of the following expressions.

(a) 1^{3i} **(b)** $\left(1 - i\sqrt{3}\right)^{1+\pi i}$ **(c)** $\left(3 + i\sqrt{3}\right)^{5i}$

2.20. Let z be a real number x in the definition for $\arctan z$ given in Section 2.5.9. Show that the expression collapses to all the values of the multivalued real arctangent function. [Hint: Write $(i + x)/(i - x)$ as $(1 - ix)/(1 + ix)$ and define $\rho e^{i\phi}$ as the polar form of $1 + ix$ with $-\pi < \phi \le \pi$.]

2.21. Find all roots of the given equation.

(a) $\sinh z = 0$ **(b)** $\cosh z = 0$ **(c)** $\tanh z = i$

(d) $\cos z = 2$ **(e)** $\sin z = \cosh 4$ **(f)** $\cot z = \dfrac{2\sqrt{3} + 3i}{7}$

2.22. Develop the expression for $w = \operatorname{arcsinh} z$ given in (2.15).

CHAPTER 3

Differentiation

3.1 THE DERIVATIVE

The definition of the derivative of a function f of a complex variable is analogous to the real case.

Definition 3.1 Derivative. If a single-valued function f is defined on a neighborhood of $z \in \mathbb{C}$, then the **derivative** of f at z is

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z},$$

provided the limit exists. If it does, we say f is **differentiable** at z.

The key feature about the above definition is that the limit must exist *no matter how* Δz approaches zero. This is why the definition insists that f be defined in a neighborhood of z. The fact that the limit must be the same regardless from which direction Δz approaches zero has strong implications, which we shall see.

Higher-order derivatives are denoted f'', f''' or $f^{(n)}$, for the second, third, and nth derivative, respectively. As with real functions the other standard way of denoting derivatives is

$$\frac{df}{dz}, \qquad \frac{d^2 f}{dz^2}, \qquad \frac{d^3 f}{dz^3}, \qquad \text{and} \qquad \frac{d^n f}{dz^n}.$$

Since the definition is essentially the same as for the real case, all of the differentiation rules based on this limit definition, of which you are familiar, still hold.

Example 3.2 From the definition, compute the derivative of $f(z) = z^n$ for $n \in \mathbb{Z}$.

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{(z + \Delta z)^n - z^n}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\binom{n}{0} z^n + \binom{n}{1} z^{n-1} \Delta z + \binom{n}{2} z^{n-2} \Delta z^2 + \cdots + \binom{n}{n-1} z \Delta z^{n-1} + \binom{n}{n} \Delta z^n - z^n}{\Delta z} \\ &= \lim_{\Delta z \to 0} \left[n z^{n-1} + \binom{n}{2} z^{n-2} \Delta z + \cdots + n z \Delta z^{n-2} + \Delta z^{n-1} \right] = n z^{n-1}. \end{aligned}$$

Other differentiation rules based on this limit definition that are still valid include:

$$(f+g)' = f' + g', \qquad (cf)' = cf', \qquad (fg)' = f'g + fg',$$
$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \qquad f(g(z))' = f'(g(z))g'(z),$$

where f and g are differentiable functions, and c is a constant. Using the series representation of e^z, the fact that $e^{z+\Delta z} = e^z e^{\Delta z}$, and the definition of the derivative, it is also easy to show that $(e^z)' = e^z$. Then, using the definitions of the trigonometric and hyperbolic functions it is not difficult to establish the following derivatives:

$$\frac{d}{dz}\sin z = \cos z, \qquad \frac{d}{dz}\cos z = -\sin z, \qquad \frac{d}{dz}\tan z = \sec^2 z,$$
$$\frac{d}{dz}\sinh z = \cosh z, \qquad \frac{d}{dz}\cosh z = \sinh z, \qquad \frac{d}{dz}\tanh z = \operatorname{sech}^2 z.$$

Finally, the rule for the derivative of an inverse is also the same as the real case. Let $w = f(z)$ and suppose both f and its inverse are differentiable. Then $z = f^{-1}(w)$, where here we assume the inverse is a single-valued function (if the inverse is multivalued, then take any one branch of that function). If we differentiate both sides of this equation with respect to z we get

$$1 = \frac{d}{dw}f^{-1}(w)\cdot\frac{dw}{dz} \qquad \Longrightarrow \qquad \frac{d}{dw}f^{-1}(w) = \frac{1}{\dfrac{df}{dz}}.$$

Since any one branch of $\log w$ is the inverse of $w = e^z$, if follows from the above that

$$\frac{d}{dw}\log w = \frac{1}{w}.$$

L'Hopital's rule also continues to hold for complex functions, that is, if f and g are differentiable and if in the limit as $z \to z_0$ the ratio $f(z)/g(z)$ is one of the indeterminate forms $0/0$ or ∞/∞, then

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \lim_{z\to z_0}\frac{f'(z)}{g'(z)}.$$

Up to this point it appears that complex derivatives are pretty much the same as real derivatives. But consider now the function $f(z) = |z|^2 = x^2 + y^2 = z\overline{z}$. This function looks very well behaved. Its real part is a nice smooth function of x and y (the partial derivatives with respect to x and y exist and are continuous everywhere) and its imaginary part is always the constant zero. If we restrict ourselves to the real line, the function is just $f = x^2$, which is a real differentiable function. However, is the complex function differentiable by the above definition?

$$L = \lim_{\Delta z\to 0}\frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z\to 0}\frac{(z+\Delta z)(\overline{z}+\overline{\Delta z}) - z\overline{z}}{\Delta z} = \lim_{\Delta z\to 0}\left(\overline{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z}\right).$$

Now, for any number $w = \rho e^{i\phi}$ we have

$$\frac{\overline{w}}{w} = \frac{\rho e^{-i\phi}}{\rho e^{i\phi}} = e^{-i2\phi},$$

therefore

$$L = \lim_{\Delta z \to 0} \left(\overline{z} + \overline{\Delta z} + z e^{-i2\arg(\Delta z)} \right).$$

But since Δz may approach 0 in any direction, the value of $e^{-i2\arg(\Delta z)}$ may be any value on the unit circle. Since this factor is multiplied by z, if $z = 0$ we conclude that $L = \overline{z}$, but if $z \neq 0$, then the limit L does not exist. We conclude that $f(z) = |z|^2$ is differentiable at $z = 0$ but not anywhere else in the complex plane. Here we see a manifestation of the strong requirement that the limit in the definition of the derivative must be the same value regardless of how Δz approaches zero. To be differentiable, it is clearly not enough for a function's real and imaginary parts to be smooth functions of x and y.

3.1.1 EXERCISES

3.1. **(a)** Use the definition of the derivative, the series definition of e^z, and the fact that $e^{a+b} = e^a e^b$ for any $a, b \in \mathbb{C}$ to prove that $\frac{de^z}{dz} = e^z$.

(b) Verify the following derivatives by first replacing the function to be differentiated by its definition in terms of the exponential function and then using the result of (a).

$$\text{(i)}\ \frac{d \sin z}{dz} = \cos z, \qquad \text{(ii)}\ \frac{d \cosh z}{dz} = \sinh z, \qquad \text{(iii)}\ \frac{d \tan z}{dz} = \sec^2 z.$$

3.2. From the definition of the derivative, show that the function $f(z) = \mathrm{Re}(z)$ is not differentiable anywhere.

3.3. Show that on any particular branch of arcsin and arctan

$$\textbf{(a)}\ \frac{d \arcsin z}{dz} = \frac{1}{(1 - z^2)^{1/2}}, \qquad \textbf{(b)}\ \frac{d \arctan z}{dz} = \frac{1}{1 + z^2}.$$

3.2 GEOMETRIC INTERPRETATION OF THE DERIVATIVE

Recall that for a real function $f(x)$, the real derivative of f at x_0 is the slope of the tangent line at that point. The tangent line is the best linear approximation to the function at x_0. Indeed, for x near x_0, the first two terms of a Taylor series gives the approximation

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0),$$

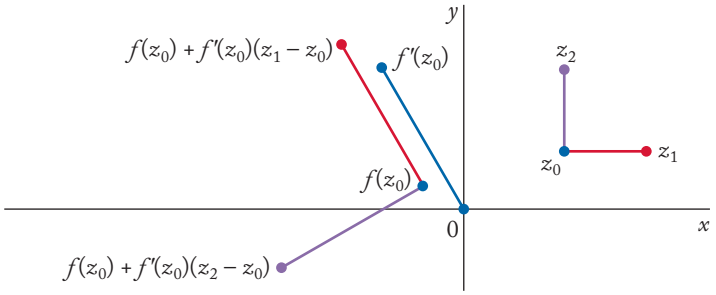


Figure 3.1: The derivative of $f(z) = \frac{1}{10}z^5$ at z_0 both scales and rotates deviations from z_0.

where the right-hand side is an expression for the tangent line. The factor $(x - x_0)$ is the deviation away from x_0. One way of interpreting the derivative of f at x_0 is to realize that it is the factor by which one must multiply the deviation from x_0 to obtain the (approximate) change in the value of the function f. Thus, for example, if $f'(x_0)$ is a large negative number, then a small positive deviation from x_0 will result in a large decrease in the function value. This same interpretation applies in the complex case. For a complex function $f(z)$ we have the approximation

$$f(z) \approx f(z_0) + f'(z_0)(z - z_0),$$

and $f'(z_0)$ is the factor that is multiplied by the deviation away from z_0 to obtain the approximate change in the function value. The difference between the complex case and the real case is that now the value of f' is a complex number and, most importantly, multiplication both scales and rotates the deviation. Thus, if $f'(z_0) = \rho e^{i\phi}$ then a deviation $(z - z_0)$ will be scaled by ρ and rotated by ϕ to give the approximate change in the function value. For this reason, in his text Needham [1997], calls the derivative an "amplitwist."

Example 3.3 Illustrate how small deviations in x and y from $z_0 = \sqrt{2}e^{i\pi/6}$ will affect $f(z) = \frac{1}{10}z^5$.

The derivative of f is $f'(z) = \frac{1}{2}z^4$, so $f(z_0) = \frac{2\sqrt{2}}{5}e^{i5\pi/6}$ and $f'(z_0) = 2e^{i2\pi/3}$. Figure 3.1 illustrates the situation. A small deviation to the right from z_0 to z_1 will result in a change in f whose modulus is 2 times larger and whose direction is rotated by $2\pi/3$. Similarly, a small deviation upward from z_0 to z_2 will cause a change in f whose modulus is 2 times larger than the deviation in z, and the change in f will be in the direction $2\pi/3 + \pi/2 = 7\pi/6$.

3.2.1 EXERCISES

3.4. Consider the mapping $f(z) = e^{i2z}$ and the point $z_0 = (\pi - 4i)/8$. Compute and plot $f(z_0)$ and $f'(z_0)$. Let δ be a small positive number. If z_0 is shifted by δ in the positive x-direction, approximately how will $f(z_0)$ change? Indicate the approximate location of $f(z_0 + \delta)$ on your plot. Do the same if z_0 is shifted by δ in the positive y-direction.

3.5. Suppose f has a derivative at z_0. If a small deviation Δz_1 is added to z_0 then $f(z_0)$ will change by an amount w_1, approximated by $w_1 \approx f'(z_0)\Delta z_1$. Now consider a different small deviation Δz_2 that is added to z_0 and the associated change in f denoted by w_2. Give an argument as to why the angle between w_1 and w_2 is the same as that between Δz_1 and Δz_2.

3.3 THE CAUCHY–RIEMANN EQUATIONS

In this section we will establish some necessary conditions for a function to be differentiable; these are called the Cauchy–Riemann equations. [1] Let $z = x + iy$ and write the function f as $f(z) = u(x, y) + iv(z, y)$, where u and v are real-valued functions defined on $\mathbb{R}^2$. Then, using the definition of the derivative of f we have

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\ &= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\,\Delta y}. \end{aligned} \tag{3.1}$$

Now suppose we choose to allow Δz to approach 0 along the real axis, that is $\Delta y \equiv 0$ and $\Delta z = \Delta x$. Then (3.1) becomes

$$\begin{aligned} f'(z) &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y) + i\,[v(x + \Delta x, y) - v(x, y)]}{\Delta x} = u_x(x, y) + iv_y(x, y), \end{aligned}$$

where the subscripts on u and v denote partial derivatives, that is $u_x = \dfrac{\partial u}{\partial x}$, etc. Now, since by definition, the limit only exists if it is the same value regardless of the manner in which Δz approaches zero, we need to obtain the same value if we allow Δz to approach 0 along the imaginary axis, that is, $\Delta x \equiv 0$ and $\Delta z = i\,\Delta y$. Note the factor of i here; it is important. In this

[1] Augustin-Louis Cauchy (1789–1857) was a French mathematician of many accomplishments including being the primary founder of complex analysis.

case, (3.1) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y) + i\,[v(x,y+\Delta y) - v(x,y)]}{i\,\Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{v(x,y+\Delta y) - v(x,y) - i\,[u(x,y+\Delta y) - u(x,y)]}{\Delta y} = v_y(x,y) - i u_y(x,y).$$

Comparing these two different ways of allowing Δz to approach zero we conclude that if the derivative exists, it is *necessary* that $u_x = v_y$ and $u_y = -v_x$. These are the *Cauchy–Riemann equations*, or just the "CR equations" for short. We have thus proved the following theorem.

Theorem 3.4 The Cauchy–Riemann (CR) Equations. *Let $z = z + iy$ and let $f(z) = u(x,y) + iv(x,y)$ where u and v are real-valued functions on $\mathbb{R}^2$. For f to be differentiable it is necessary that*

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x. \tag{3.2}$$

The other thing we can conclude is that if the derivative exists then both $u_x + iv_x$ and $v_y - iu_y$ are valid expressions for f'. Let us return to the example of $f(z) = |z|^2$. In this case we have $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$. It follows that

$$u_x = 2x, \qquad u_y = 2y, \qquad v_x = 0, \qquad v_y = 0.$$

Clearly the only location at which the CR equations hold is at $(x,y) = (0,0)$.

Example 3.5 Determine where the function $f(z) = x^2 - 6y + i2xy$ satisfies the CR equations.

We have $u = x^2 - 6y$ and $v = 2xy$; thus

$$u_x = 2x, \qquad u_y = -6, \qquad v_x = 2y, \qquad v_y = 2x.$$

It is easy to see that $u_x = v_y$ everywhere, but for $u_y = -v_x$ to be satisfied we require $y = 3$. Therefore the only places where f might be differentiable is along the line $y = 3$ in the complex plane.

Example 3.6 Show that the function $f(z) = x^2 + x - y^2 + i(2xy + y)$ satisfies the CR equations everywhere.

We have $u = x^2 + x - y^2$ and $v = 2xy + y$ so

$$u_x = 2x + 1, \qquad u_y = -2y, \qquad v_x = 2y, \qquad v_y = 2x + 1.$$

Clearly the CR equations hold for all x, y.

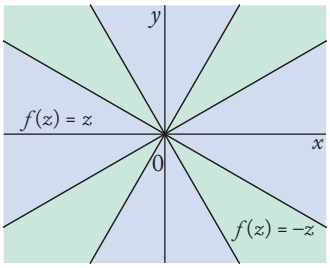


Figure 3.2: The domain of the function in Example 3.7.

Is conjugation a differentiable function? Let $f(z) = \overline{z} = x - iy$. This means $u(x, y) = x$ and $v(x, y) = -y$, therefore

$$u_x = 1, \qquad u_y = 0, \qquad v_x = 0, \qquad v_y = -1.$$

The CR equation $u_y = -v_x$ is clearly satisfied but the equation $u_x = v_y$ is not satisfied. We conclude that f is not differentiable anywhere, therefore conjugation is not a differentiable operation.

We emphasize that the CR equations are only a necessary condition for differentiability. Just because the CR equations hold at some point does not imply that the function is differentiable there, as illustrated in the next example.

Example 3.7 Consider the function

$$f(z) = \begin{cases} -z & \text{if } \frac{\pi}{6} < |\text{Arg}(z)| < \frac{\pi}{3} \text{ or } \frac{2\pi}{3} < |\text{Arg}(z)| < \frac{5\pi}{6}, \text{ and } z \neq 0, \\ z & \text{otherwise.} \end{cases}$$

In the interior of the blue colored region in Fig. 3.2, we have $f(z) = z$ so $u = x$ and $v = y$. Hence $u_x = 1$, $u_y = 0$, $v_x = 0$, and $v_y = 1$ and the CR equations hold. Similarly, in the interior of the green colored region we have $f(z) = -z$ and the CR equations hold. At the boundaries between the two regions the CR equations do not even make sense because the partial derivatives of u and v do not exist since f is discontinuous. However, at the origin the partial derivatives of u and v do exist since f is defined to be z at the origin and along both the real and imaginary axes. Further, the CR equations hold at the origin. Nonetheless, the function cannot be differentiable at the origin because as Δz approaches 0 along the real axis the limit is 1 while it is -1 if Δz approaches 0 along the ray $\theta = \pi/4$.

3.3.1 SUFFICIENT CONDITIONS FOR DIFFERENTIABILITY

In order to establish a set of sufficient conditions for differentiability we simply need to impose continuity of the partial derivatives of u and v along with the CR equations.

Theorem 3.8 Sufficient Conditions for Differentiability. *Let* $f(z) = u(x, y) + iv(x, y)$ *be defined in a neighborhood* $N_\delta(z_0)$ *of* $z_0 = x_0 + iy_0$. *If the first partial derivatives of* u *and* v *exist and are continuous in* $N_\delta(z_0)$ *and if they satisfy the CR equations at* (x_0, y_0), *then* f *is differentiable at* z_0.

Proof. To prove the theorem we need to show that the limit

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists. First recall that if a real function g is continuous at x, that is if $\lim_{h \to 0} g(x + h) = g(x)$, we may write

$$g(x + h) = g(x) + \epsilon \qquad \text{where } \epsilon \to 0 \text{ as } h \to 0. \tag{3.3}$$

In particular, if p is a real continuously differentiable function at x, then we have

$$\frac{dp}{dx}(x) = \lim_{h \to 0} \frac{p(x + h) - p(x)}{h}$$

is equivalent to

$$p(x + h) - p(x) = \left(\frac{dp}{dx}(x) + \epsilon \right) h, \tag{3.4}$$

where $\epsilon \to 0$ as $h \to 0$.

Let $w = f(z)$ and define $\Delta w = f(z_0 + \Delta z) - f(z_0)$. Assume that Δz is sufficiently small so that $z_0 + \Delta z$ is in the neighborhood $N_\delta(z_0)$. Then

$$\Delta w = u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0).$$

Adding and subtracting some like terms and applying the result (3.4), which is valid since u and v have continuous partial derivatives in $N_\delta(z_0)$, we have

$$\begin{aligned}
\Delta w &= u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y) + u(x_0, y_0 + \Delta y) - u(x_0, y_0) \\
&\quad + i\left[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0 + \Delta y) + v(x_0, y_0 + \Delta y) - v(x_0, y_0)\right] \\
&= \left(\frac{\partial u}{\partial x}(x_0, y_0 + \Delta y) + \epsilon_1 \right) \Delta x + \left(\frac{\partial u}{\partial y}(x_0, y_0) + \epsilon_2 \right) \Delta y \\
&\quad + i \left[\left(\frac{\partial v}{\partial x}(x_0, y_0 + \Delta y) + \epsilon_3 \right) \Delta x + \left(\frac{\partial v}{\partial y}(x_0, y_0) + \epsilon_4 \right) \Delta y \right],
\end{aligned}$$

where $\epsilon_j \to 0$ as $(\Delta x, \Delta y) \to (0,0)$, $1 \le j \le 4$. Now using the fact that u_x and v_x are continuous at (x_0, y_0), we may use the result (3.3) to write

$$\Delta w = \left(\frac{\partial u}{\partial x}(x_0, y_0) + \epsilon_5 + \epsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y}(x_0, y_0) + \epsilon_2\right)\Delta y + i\left[\left(\frac{\partial v}{\partial x}(x_0, y_0) + \epsilon_6 + \epsilon_3\right)\Delta x + \left(\frac{\partial v}{\partial y}(x_0, y_0) + \epsilon_4\right)\Delta y\right],$$

where ϵ_5 and ϵ_6 also vanish as Δy goes to zero. Finally, using the fact that the CR equations hold at (x_0, y_0) we can replace u_y and v_y in the above and collect to get

$$\Delta w = \left(\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)\right)(\Delta x + i\Delta y) + (\epsilon_1 + \epsilon_5 + i\epsilon_3 + i\epsilon_6)\Delta x + (\epsilon_2 + i\epsilon_4)\Delta y.$$

Dividing this by $\Delta z = \Delta x + i\Delta y$ and taking the limit as $\Delta z \to 0$, that is, $(\Delta x, \Delta y) \to (0,0)$, we have

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0),$$

where we have used the fact that each ϵ_j tends to zero as Δx and Δy tend to zero. Since u_x and v_x are defined at (x_0, y_0) we have shown that the limit exists. □

Consider again the function in Example 3.5. In this example the CR equations held at all points with $y = 3$. Further, the partial derivatives of u and v in that example were clearly continuous everywhere. Hence, we conclude that the function is differentiable at all points on the line $y = 3$.

3.3.2 OTHER FORMS OF THE CAUCHY–RIEMANN EQUATIONS

In this section we re-write the Cauchy–Riemann equations in terms of complex conjugate coordinates, and in terms of polar coordinates.

Recall from Section 2.3 that we may use the complex conjugate coordinates

$$z = x + iy \qquad \text{and} \qquad \overline{z} = x - iy. \tag{3.5}$$

Assuming f is differentiable, from the previous section we have two equivalent expressions for the derivative:

$$f' = u_x + iv_x \qquad \text{and} \qquad f' = v_y - iu_y.$$

Considering now f, u, and v as functions of z and $\overline{z}$ rather than x and y, we may use chain rule to write these expressions for f' as

$$f' = \left[\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial u}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial x}\right] + i\left[\frac{\partial v}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial v}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial x}\right]$$

and

$$f' = \left[\frac{\partial v}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial v}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial y}\right] - i\left[\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial u}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial y}\right].$$

Using (3.5) these become

$$f' = \left[\frac{\partial u}{\partial z} + \frac{\partial u}{\partial \overline{z}}\right] + i\left[\frac{\partial v}{\partial z} + \frac{\partial v}{\partial \overline{z}}\right] \qquad \text{and} \qquad f' = i\left[\frac{\partial v}{\partial z} - \frac{\partial v}{\partial \overline{z}}\right] + \left[\frac{\partial u}{\partial z} - \frac{\partial u}{\partial \overline{z}}\right],$$

or, equivalently,

$$f' = \frac{\partial}{\partial z}[u + iv] + \frac{\partial}{\partial \overline{z}}[u + iv] \qquad \text{and} \qquad f' = \frac{\partial}{\partial z}[u + iv] - \frac{\partial}{\partial \overline{z}}[u + iv].$$

These two expressions for f' must be the same. The first term in each of the above expressions is simply $\frac{\partial f}{\partial z}$ and the second is $\frac{\partial f}{\partial \overline{z}}$, although the sign is different. Thus, it must be zero, which gives us the conjugate coordinate form of the CR equations.

Definition 3.9 The complex conjugate coordinate form of the CR equations is

$$\frac{\partial f}{\partial \overline{z}} = 0. \tag{3.6}$$

In other words, f must be able to be written in terms of z alone; there can be no $\overline{z}$ factors in f that do not cancel out. The last phrase is important. For example, the function $f(z) = z\overline{z}$ clearly fails to satisfy (3.6). But now consider the function $f(z) = \operatorname{Log} z$. We have that

$$f(z) = \operatorname{Log} z = \ln|z|\, e^{i \operatorname{Arg}(z)} = \ln\sqrt{z\overline{z}}\, e^{i \operatorname{atan2}(y,x)} = \ln\sqrt{z\overline{z}}\, e^{i \operatorname{atan2}(-i(z-\overline{z})/2,(z+\overline{z})/2)}.$$

If one were only given the last expression on the right, one might think there is no way the $\overline{z}$ factors will cancel out and so conclude that f is not analytic. However, the $\overline{z}$ terms do in fact cancel since the expression is equal to $\operatorname{Log} z$. Further, it is not difficult to show that $\operatorname{Log} z$ satisfies the CR equations, hence indeed this function is analytic. Equation (3.6) is just another way of saying that conjugation is not differentiable, as we first saw on page 57. When $f(z)$ is written in terms of the real and imaginary parts of z, then the CR equations given by (3.2) should be used, but if $f(z)$ is written in terms of z and $\overline{z}$ then (3.6) may be more convenient; it tells us that if $\overline{z}$ appears in the expression for f, and provided that you are certain the $\overline{z}$ factors do not cancel out, then f is not differentiable.

It is also useful to express the CR equations in terms of polar coordinates r and θ. In this case we assume that f can be written as

$$f(z) = u(r,\theta) + iv(r,\theta), \qquad \text{where } z = re^{i\theta}.$$

One can develop the polar form of the CR equations for this f either by using the coordinate change between (x, y) and (r, θ), or directly from the definition of f' as we did for the CR equations in Cartesian coordinates originally. We shall do the latter here.

First suppose Δz approaches zero along a ray in the direction of z. Then we may write $\Delta z = \Delta r e^{i\theta}$ and $z + \Delta z = (r + \Delta r)e^{i\theta}$ so that

$$\begin{aligned} f' &= \lim_{\Delta r \to 0} \frac{u(r+\Delta r,\theta) + iv(r+\Delta r,\theta) - u(r,\theta) - iv(r,\theta)}{\Delta r e^{i\theta}} \\ &= e^{-i\theta} \lim_{\Delta r \to 0} \left[\frac{u(r+\Delta r,\theta) - u(r,\theta)}{\Delta r} + i \frac{v(r+\Delta r,\theta) - v(r,\theta)}{\Delta r} \right] \\ &= e^{-i\theta} \left[\frac{\partial u}{\partial r}(r,\theta) + i \frac{\partial v}{\partial r}(r,\theta) \right] = e^{-i\theta}(u_r + iv_r). \end{aligned} \tag{3.7}$$

Now consider the case where $z + \Delta z$ approaches z along the circle of radius r centered at the origin. Then we have $z + \Delta z = re^{i(\theta+\Delta\theta)}$, and

$$\Delta z = re^{i(\theta+\Delta\theta)} - re^{i\theta} = re^{i\theta}\left(e^{i\Delta\theta} - 1\right) \approx re^{i\theta} i\Delta\theta,$$

where the last approximation is valid when $|\Delta\theta|$ is small (using the first two terms of the series definition of the exponential). Then

$$\begin{aligned} f' &= \lim_{\Delta\theta \to 0} \frac{u(r,\theta+\Delta\theta) + iv(r,\theta+\Delta\theta) - u(r,\theta) - iv(r,\theta)}{re^{i\theta} i\Delta\theta} \\ &= \frac{1}{r} e^{-i\theta} \lim_{\Delta\theta \to 0} \left[\frac{v(r,\theta+\Delta\theta) - v(r,\theta)}{\Delta\theta} - i \frac{u(r,\theta+\Delta\theta) - u(r,\theta)}{\Delta\theta} \right] \\ &= \frac{1}{r} e^{-i\theta} \left[\frac{\partial v}{\partial \theta}(r,\theta) - i \frac{\partial u}{\partial \theta}(r,\theta) \right] = \frac{1}{r} e^{-i\theta}(v_\theta - iu_\theta). \end{aligned} \tag{3.8}$$

Both (3.7) and (3.8) are valid expressions for f', hence they must be equal. This gives the following polar form for the Cauchy–Riemann equations.

Definition 3.10 The polar form of the CR equations is

$$u_r = \frac{1}{r} v_\theta, \qquad \text{and} \qquad \frac{1}{r} u_\theta = -v_r.$$

Example 3.11 Determine if the CR equations hold for the function $f(z) = r^2 \sin(2\theta) - ir^2 \cos(2\theta)$ and if they hold, what is f'?

We have

$$u_r = 2r\sin(2\theta), \quad u_\theta = 2r^2\cos(2\theta), \quad v_r = -2r\cos(2\theta), \quad v_\theta = 2r^2\sin(2\theta).$$

Clearly the Cauchy–Riemann equations hold for this f, and

$$\begin{aligned} f' &= e^{-i\theta}(u_r + i v_r) = e^{-i\theta}(2r\sin(2\theta) - i2r\cos(2\theta)) \\ &= e^{-i\theta}(-i2r)(\cos(2\theta) + i\sin(2\theta)) = -i2re^{-i\theta}e^{i2\theta} = -i2re^{i\theta}. \end{aligned}$$

This last expression is equal to $-i2z$, implying that $f(z) = -iz^2$. A short computation will show that the given f is indeed equal to this.

One could also develop the CR equations in the case where f is also written in polar form rather than Cartesian form, that is, in the case $f(z) = R(r,\theta)e^{i\Theta(r,\theta)}$. This is left as an exercise.

3.3.3 EXERCISES

3.6. Let $z = x + iy = re^{i\theta}$. Determine where the following functions are differentiable.

(a) $f(z) = x^2 - y^2 + x - 1 + i(2x + y)$,

(b) $f(z) = \dfrac{x^2 + y^2 - 1 - i2y}{(x-1)^2 + y^2}$,

(c) $f(z) = 2x^2y + 4xy + i\left(2(x+1)y^2 - \dfrac{2}{3}x^3 - 3x^2\right)$,

(d) $f(z) = r^2\sin\theta - i2r^2\cos\theta$.

3.7. Suppose $f(z)$ is differentiable in a domain D. Prove that $f(z)$ must be constant in D if any of the following are constant in D.

(a) $\mathrm{Re}(f(z))$, **(b)** $\mathrm{Im}(f(z))$, **(c)** $|f(z)|$.

3.8. Verify that $f(z) = \mathrm{Log}\, z$ satisfies the polar form of the CR equations provided $z \neq 0$ and $\mathrm{Arg}(z) \neq \pi$.

3.9. Suppose $z = re^{i\theta}$ and $f(z) = R(r,\theta)e^{i\Theta(r,\theta)}$, where R and Θ are real functions of the real variables r and θ. Determine the Cauchy–Riemann equations for f in terms of partial derivatives of R and Θ. Simplify them as much as possible.

3.4 ANALYTIC FUNCTIONS

Definition 3.12 If $f'(z)$ exists at all points in a domain R then the function f is said to be **analytic in R**. Other equivalent terms are "holomorphic" and "regular."

Recall that a domain is an open connected set. A rational function $f(z) = P(z)/Q(z)$, is analytic on any domain R that does not include any points where the denominator $Q(z)$ is zero.

Definition 3.13 The function f is said to be **analytic at a point z_0** if $f'(z)$ exists in a neighborhood of z_0.

To be analytic at a point is slightly stronger than being differentiable there. To be differentiable at a point all that is required is that the conditions in Theorem 3.8 are satisfied. In contrast, to be analytic at a point the function must be differentiable at all points in a neighborhood. Consider Example 3.5 one more time. The function there was differentiable at all those points of $\mathbb{C}$ with imaginary part equal to 3. Since for any such point a neighborhood of that point must include points where $y \neq 3$, we conclude that the function is not differentiable at all points in the neighborhood and so is *not* analytic anywhere.

Definition 3.14 If f is analytic in all of $\mathbb{C}$ then f is said to be an **entire** function.

Polynomials are entire functions, as is the exponential function and therefore also cosine and sine. The function $f(z) = 1/z$ is analytic everywhere except at $z = 0$, hence it is not entire.

Definition 3.15 Any point where f is analytic is called an **ordinary** point of f.

Definition 3.16 If f is not analytic at a point z_0 but every neighborhood of z_0 contains points at which f is analytic, then z_0 is called a **singular** point of f.

In other words, a point is a singular point of f if f is not analytic at that point but there are nearby points where f is analytic. We will discuss singular points more fully in Section 3.5.

Definition 3.17 If f is analytic at z_0 and if $f'(z_0) = 0$, then z_0 is called a **critical** point of f.

For example, the critical points of $f(z) = z + e^z$ are $z = \log(-1) = i\pi(1 + 2k)$, $k \in \mathbb{Z}$.

Definition 3.18 If f is analytic in a region R containing z_0 and if there is an analytic function g on R such that f can be written in the form

$$f(z) = (z - z_0)^m g(z), \qquad \forall z \in R, \tag{3.9}$$

where $m \in \mathbb{Z}_+$ and $g(z_0) \neq 0$, then z_0 is called a **zero of order m** of f. A zero of order one is called a **simple zero**.

Assuming for now that g has at least $m - 1$ derivatives, it is easy to show that if f has a zero of order m at z_0 then f and its first $m - 1$ derivatives at z_0 are all zero. For example, $f(z) =$

$z^2 + 2z + 1 = (z + 1)^2$ has a zero of order 2 at $z_0 = -1$, and both f and f' are zero at z_0. In contrast, $f(z) = z^2 + 4$ has simple zeros at $z_0 = \pm 2i$ and while f is zero at these points, $f' = 2x$ is not. Definition 6.36 in Section 6.3 is an alternative definition of zeros for f based on the first $m - 1$ derivatives being zero. We show there that both definitions are equivalent.

The definition of a zero for f implies that that zero is isolated.

Theorem 3.19 Zeros of Order m are Isolated. *If f is analytic in a region R containing z_0 and z_0 is a zero of order m for f, then there is a deleted neighborhood of z_0 on which f is nonzero.*

Proof. By Definition 3.18 there is an analytic function g in R such that (3.9) holds and $g(z_0) \neq 0$. Since g is analytic it must be continuous at z_0 and so by continuity it must be nonzero in some neighborhood $N_\delta(z_0) \subset R$. But the function $(z - z_0)^m$ is nonzero except at z_0. Hence, f is nonzero in the deleted neighborhood $N_\delta(z_0) \setminus \{z_0\}$. □

You might wonder whether it is possible for an analytic function f to be equal to zero at some point z_0 without there existing an analytic function g that satisfies Definition 3.18. The answer is no, unless the function f is identically zero in R. We shall prove this in Theorem 6.37. The important point is that zeros of an analytic function f are isolated (unless f is identically zero).

Equation (3.9) may be re-arranged to say

$$\lim_{z \to z_0} \frac{f(z)}{(z - z_0)^m} = g(z_0) = A \neq 0. \tag{3.10}$$

From this it follows that as z approaches z_0, a zero of order m for f, the modulus $|f(z)|$ approaches zero like $|A|\,|z - z_0|^m$. Likewise, the argument of f behaves like $\arg A + m \arg(z - z_0)$.

3.4.1 INVERTIBILITY

Analytic functions are locally invertible at any ordinary point z_0 that is not a critical point. The adjective "locally" here means that the size of the neighborhood of $w_0 = f(z_0)$ in which an inverse function exists may be small. For example, the function $w = f(z) = e^z$ is an entire function and has no critical points since its derivative is never zero. Suppose $z_0 = 0$. An analytic inverse for f defined locally to $w_0 = f(z_0) = 1$ is any appropriate branch of the function $g(w) = \log w$. Any such branch is not defined at $w = 0$ so the largest δ neighborhood of w_0 in which the inverse is defined is one with $\delta = 1$.

Theorem 3.20 Local Invertibility of Analytic Functions. *If f is analytic at z_0 and if z_0 is not a critical point of f then there exists an analytic inverse of f defined on a neighborhood $N_\epsilon(f(z_0))$ for some $\epsilon > 0$.*

Proof. Since f is analytic at z_0 it must be analytic in some neighborhood $N_{\delta_1}(z_0)$. Further, since z_0 is not a critical point, $f'(z_0) \neq 0$, and by continuity of f there must be a neighborhood $N_{\delta_2}(z_0)$ in which $f'(z) \neq 0$. Choose $\delta = \min(\delta_1, \delta_2)$.

Write $f(z) = U(x, y) + iV(x, y)$ and $z_0 = x_0 + iy_0$. The equations $u = U(x, y)$, $v = V(x, y)$ are a transformation from $N_\delta((x_0, y_0)) \subset \mathbb{R}^2$ to the (u, v)-plane and the Jacobian of this transformation is

$$J = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix} = U_x V_y - V_x U_y = U_x^2 + U_y^2,$$

where the last equality is due to the fact that since f is analytic, the Cauchy–Riemann equations hold in $N_\delta((x_0, y_0))$. Recalling that $f'(z) = U_x + iU_y$ we see that $J = |f'(z)|^2$, which means that J is nonzero in $N_\delta((x_0, y_0))$. Let $u_0 = U(x_0, y_0)$ and $v_0 = V(x_0, y_0)$. Because the Jacobian of the transformation is nonzero, it follows from the inverse function theorem (see, for example, Taylor and Mann [1983]) that there exists a unique inverse transformation, $x = X(u, v)$, $y = Y(u, v)$, defined on some neighborhood $N_\epsilon(u_0, v_0)$, $\epsilon > 0$, such that

$$u = U(X(u, v), Y(u, v)), \quad \text{and} \quad v = V(X(u, v), Y(u, v)), \qquad \forall (u, v) \in N_\epsilon((x_0, y_0)).$$

Defining $w = u + iv$, $w_0 = u_0 + iv_0$, and $g(w) = X(u, v) + iY(u, v)$, it follows that $w = f(g(w))$ in $N_\epsilon(w_0)$. Further, the inverse function theorem guarantees that the first partial derivatives of X and Y are continuous and satisfy

$$X_u = \frac{v_y}{J}, \qquad X_v = \frac{-u_y}{J}, \qquad Y_u = \frac{-v_x}{J}, \qquad Y_v = \frac{u_x}{J},$$

in $N_\epsilon((u_0, v_0))$. From this it is clear that g satisfies the CR equations in $N_\epsilon(w_0)$, and since the partial derivatives of X and Y are continuous g satisfies the sufficient conditions for differentiability. Hence g is analytic in $N_\epsilon(w_0)$. □

3.4.2 HARMONIC FUNCTIONS

Now suppose $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D and assume u and v have continuous second partial derivatives in D (we show that they always will in Section 5.2). Then since the CR equations hold in D we have

$$\begin{aligned} u_{xx} &= (v_y)_x && \text{since } u_x = v_y \\ &= (v_x)_y && \text{changing the order of differentiation} \\ &= (-u_y)_y && \text{since } u_y = -v_x. \end{aligned}$$

Thus,

$$u_{xx} + u_{yy} = 0. \tag{3.11}$$

In the same we we can show that $v_{xx} + v_{yy} = 0$. Equation (3.11) is called Laplace's equation in two dimensions.[2] It is left as an exercise to show that u also satisfies

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \tag{3.12}$$

which is the polar form of Laplace's equation. Laplace's equation has a large number of important applications. It is the governing equation for the steady-state temperature of an object, it dictates the electrostatic potential due to an array of charges exterior to the domain, and it determines the velocity potential in two-dimensional fluid flow.

Definition 3.21 Any solution u of Eq. (3.11) is called a **harmonic** function.

The above discussion has proved the following theorem.

Theorem 3.22 *If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then both u and v are harmonic functions in D.*

There are lots of harmonic functions, for example, $u(x, y) = Ax + By + C$, $u(x, y) = A(x^2 - y^2)$, $u(x, y) = e^x \cos y$, or $u(r, \theta) = r^A \sin(A\theta)$. However, for an analytic function f, its real and imaginary parts are related by the CR equations hence they cannot be two completely independent harmonic functions. Indeed, if $f = u + iv$ is analytic and u is specified, then that determines v up to a constant. We illustrate with an example.

Example 3.23 Let $u(x, y) = 3x^2y - y^3$. First show that u is a harmonic function, then find v such that $f = u + iv$ is an analytic function.

We have

$$u_x = 6xy, \qquad u_y = 3x^2 - 3y^2, \qquad u_{xx} = 6y, \qquad u_{yy} = -6y.$$

Therefore, $u_{xx} + u_{yy} = 0$ so u is a harmonic function. Now the first CR equation gives

$$v_y = u_x = 6xy \qquad \Longrightarrow \qquad v = 3xy^2 + h(x),$$

where $h(x)$ is some arbitrary function of x. Differentiating this expression with respect to x and using the second CR equation we have

$$\begin{aligned} v_x = 3y^2 + h'(x) = -u_y = -3x^2 + 3y^2 \\ \Longrightarrow \; h'(x) = -3x^2 \quad \Longrightarrow \quad h(x) = -x^3 + K, \end{aligned}$$

[2] Pierre-Simon Laplace (1749–1827) was a French mathematician who contributed in many areas including astronomy, engineering, physics, statistics, and philosophy.

where K is an arbitrary constant. We conclude that $v(x, y) = 3xy^2 - x^3 + K$, and $f = u + iv$ is analytic.

For $f = u + iv$, since v is completely determined (up to a constant) by u, there is special terminology for v.

Definition 3.24 If $f(z) = u(x, y) + iv(x, y)$ is analytic, then v is called the **harmonic conjugate** of u.

This relationship between u and v is not symmetric; u is *not* the harmonic conjugate of v. This can be seen by considering $g = -if = v - iu$. Clearly since f is analytic, so is g and, hence, from the definition we see that $-u$ is the harmonic conjugate of v.

Do harmonic conjugates always exist? The answer is yes, which we shall prove in Theorem 5.24. Thus, every harmonic function u is the real part of some analytic function $f = u + iv$.

3.4.3 EXERCISES

3.10. Let $z = x + iy$, where x and y are real. Determine where the following functions are differentiable and where they are analytic.

(a) $x^2 - 3x - y^2 + i(2xy - 3y + 4)$, **(b)** $x \sin y + ix \cos y$.

3.11. Let $f(z) = r^2 \sin(2\theta) + iv(r, \theta)$ where $z = re^{i\theta}$. Assuming f is analytic, determine $v(r, \theta)$ and find the derivative of f.

3.12. Show that if $f(r, \theta) = u(r, \theta) + iv(r, \theta)$ is analytic, then u satisfies Eq. (3.12). Show that v satisfies the same equation.

3.13. Verify directly that the real and imaginary parts of the following analytic functions satisfy Laplace's equation.

(a) $Az^2 + Bz + C$, $A, B, C \in \mathbb{C}$, **(b)** $\frac{1}{z}$, **(c)** e^z.

3.14. Find harmonic conjugates of the following functions.

(a) $u = e^x \sin y$, **(b)** $u = xy + x - y$, **(c)** $u = \ln|z|$, for $-\pi < \theta < \pi$.

[Hint: For (c) it is easiest to use the polar form of the CR equations.]

3.5 SINGULAR POINTS

As indicated in Definition 3.16, singular points are points where a function f fails to be analytic but in a neighborhood of which there are always some points where f is analytic.

Before we discuss the different classifications of singularities we describe how to deal with singularities at infinity. Given a function $f(z)$, let $w = 1/z$ and define $g(w) = f(1/w)$. We then say that f has a singularity at ∞ if g has a singularity at 0. For example, $f(z) = z^2$, has a singularity at $z = \infty$ because $g(w) = 1/w^2$ has a singularity at $w = 0$.

Singular points may be isolated or non-isolated. These terms mean exactly what you think they would mean; the precise definition is:

Definition 3.25 A singular point z_0 of f is **isolated** if there exists a deleted neighborhood N of z_0 such that there are no singularities of f in N. Any singularity that is not isolated is called **non-isolated**.

For the point at infinity, a deleted neighborhood is the region $|z| > R$ for some large R; it is perhaps best to think of stereographic projection in which case a deleted neighborhood of the point at infinity is the stereographic projection of a deleted neighborhood of the north pole on the Riemann sphere.

3.5.1 ISOLATED SINGULARITIES

There are three types of isolated singularities: poles, removable singularities, and essential singularities. The standard definitions for these three types require an understanding of Laurent series, which we cover in Chapter 6. The definitions we give here are equivalent and provide (in two cases) a test for determining the type. We will revisit the definition of these singularities in Section 6.5.

Poles

Definition 3.26 Let z_0 be an isolated singularity of f. If there is a neighborhood N of z_0 and an analytic function g on N such that f can be written in the form

$$f(z) = \frac{g(z)}{(z - z_0)^m}, \qquad \forall z \in N \setminus \{z_0\}, \tag{3.13}$$

where $m \in \mathbb{Z}_+$, and $g(z_0) \neq 0$, then z_0 is called a **pole of order m**. A pole of order one is called a **simple pole**.

Equation (3.13) can be re-arranged to say

$$\lim_{z \to z_0} (z - z_0)^m f(z) = A \neq 0. \tag{3.14}$$

Poles are, in one sense, the inverses of zeros. The above condition should be compared with the condition for a zero given by (3.10). The inverse nature of poles and zeros is reflected in the fact that a zero of order m for $f(z)$ is a pole of order m for $h(z) = \frac{1}{f(z)}$. This is true because, if z_0 is a zero of order m for f, there exists a nonzero constant $A \in \mathbb{C}$ such that

$$\lim_{z\to z_0} (z-z_0)^m h(z) = \lim_{z\to z_0} (z-z_0)^m \frac{1}{f(z)} = \frac{1}{\lim_{z\to z_0} \frac{f(z)}{(z-z_0)^m}} = \frac{1}{A} \neq 0.$$

The modulus of $f(z)$ near a pole of order m grows like $|A| \,/\, |z-z_0|^m$ and its argument behaves like $\arg(A) - m\arg(z-z_0)$. As one approaches a pole of f, the limiting value is always the point at infinity.

Example 3.27 Investigate the poles of the following functions and illustrate with phase plots.

$$\text{(a) } f(z) = \frac{1}{(z-2)^4}, \qquad \text{(b) } f(z) = \frac{2z-3i}{(z-1)(z+3)^2},$$

$$\text{(c) } f(z) = (z+i)^3, \qquad \text{(d) } f(z) = \frac{e^z-1}{z^2(z+4i)}.$$

(a) $f(z) = \dfrac{1}{(z-2)^4}$ has a pole of order four at $z_0 = 2$. Figure 3.3a shows a phase plot. Note that the colors go through four cycles in reverse order as you go around z_0.

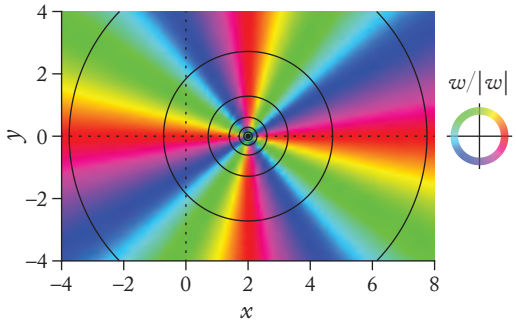
(b) $f(z) = \dfrac{2z-3i}{(z-1)(z+3)^2}$ has a simple pole at $z_0 = 1$, a pole of order two at $z_0 = -3$, and a simple zero at $z_0 = 3i/2$. The phase plot in Fig. 3.3b shows all three of these features. The colors go in normal order around the zero at $3i/2$ but go in reverse order around the poles; the colors go through two cycles around the pole at $z_0 = -3$.

(c) $f(z) = (z+i)^3$ has a pole of order 3 at $z = \infty$ since $g(w) = f(1/w) = \left(\frac{1}{w} + i\right)^3 = \left(\frac{1+iw}{w}\right)^3$ has a pole of order 3 at $w = 0$. Figure 3.3c shows a phase plot for g where clearly the colors go through three cycles in reverse order around $w = 0$.

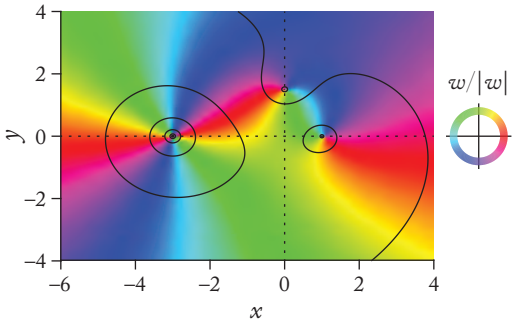
(d) $f(z) = \dfrac{e^z-1}{z^2(z+4i)}$ has a simple pole at $z_0 = -4i$, but, unlike you might expect, $z_0 = 0$ is *not* a pole of order two because

$$\lim_{z\to 0} (z-0)^2 f(z) = \lim_{z\to 0} \frac{e^z-1}{z+4i} = 0.$$

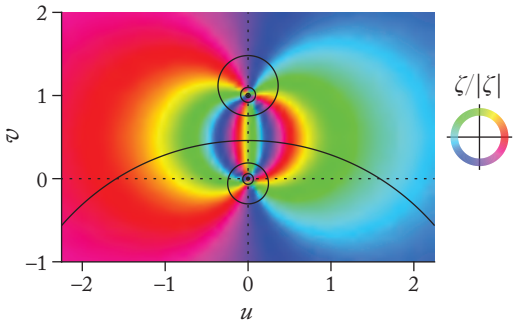
Since the limit is zero, our definition tells us that $z_0 = 0$ cannot be a pole of order two. Yet $z_0 = 0$ is clearly a singularity of f. It is possible that it is not a pole, but if it is actually a pole, then the fact that using $n = 2$ gave us a limit of zero tells us that the



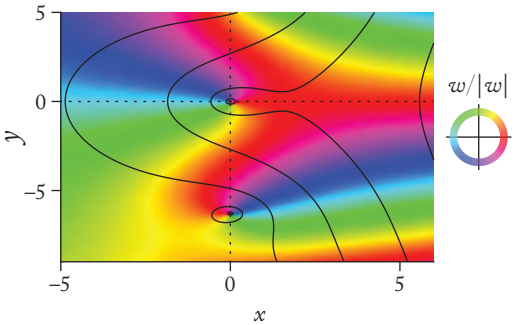
(a) The function $w = f(z) = \dfrac{1}{(z-2)^4}$



(b) The function $w = f(z) = \dfrac{3z-2}{(z-1)(z+3)^2}$



(c) The function $\zeta = g(w) = \dfrac{1+iw}{w}$



(d) The function $w = f(z) = \dfrac{e^z - 1}{z^2(z+4i)}$

Figure 3.3: Phase plots of the functions from Example 3.27.

order must be smaller than two. Indeed, if we choose $n = 1$ we have, using L'Hopital's rule,

$$\lim_{z\to 0}(z-0)\,f(z) = \lim_{z\to 0}\frac{e^z-1}{z(z+4i)} = \lim_{z\to 0}\frac{e^z}{2z+4i} = \frac{1}{4i} \neq 0.$$

Thus, we see that $z = 0$ is a simple pole for f. From Fig. 3.3d we see that indeed the colors only go through one cycle in reverse order around $z_0 = -4i$ and around $z_0 = 0$. The simple zero for f at $z_0 = -2\pi i$ is also evident on the figure.

Definition 3.28 A function that is analytic in a domain D except for a finite number of poles is called a **meromorphic** function.

Removable Singularities

Definition 3.29 Let z_0 be an isolated singularity of f. If there is a constant $A \in \mathbb{C}$ such that

$$\lim_{z \to z_0} f(z) = A,$$

then we say z_0 is a **removable singularity** of f.

In this definition A may be zero, but, since it is restricted to lie in $\mathbb{C}$, it cannot be the point at infinity. (The point at infinity is what is obtained for this limit if z_0 is a pole.)

A function with a removable singularity at z_0 can be defined at z_0 so that it becomes analytic there. For example, the function

$$f(z) = \frac{z^3 + 8z^2 + 17z + 10}{z + 1},$$

clearly has a singularity at $z = -1$ since the denominator is zero there. However, it is not a simple pole since

$$\lim_{z \to -1} (z - (-1)) \frac{z^3 + 8z^2 + 17z + 10}{z + 1} = \lim_{z \to -1} (z^3 + 8z^2 + 17z + 10) = 0.$$

To test if the singularity is removable, using L'Hopital's rule we have

$$\lim_{z \to -1} f(z) = \lim_{z \to -1} \frac{z^3 + 8z^2 + 17z + 10}{z + 1} = \lim_{z \to -1} \frac{3z^2 + 16z + 17}{1} = 4.$$

Thus, by the definition, the point $z_0 = -1$ is a removable singularity of f. Indeed, it is not difficult to show that $f(z)$ can be written

$$f(z) = \frac{(z + 1)(z + 2)(z + 5)}{z + 1}.$$

Even though the factor $(z + 1)$ cancels, the function f, as originally written, still has a singularity at $z = -1$, since f is not defined there. However, one can always define a new function $F(z) = (z + 2)(z + 5)$. This new function is equal to $f(z)$ everywhere that f is defined, plus the new function is defined and analytic at $z = -1$. Thus, this definition has removed the singularity.

Example 3.30 Find and classify the singularities of the function $f(z) = \dfrac{2iz + \pi}{\cos(iz)}$. If any singularities are removable, define an analytic function that is equal to f but has those singularities removed.

Since polynomials and the cosine functions are analytic, the only singularities of f will be when the denominator is zero.

$$\cos(iz) = 0 \qquad \Longrightarrow \qquad z = -i\left(\frac{\pi}{2} + k\pi\right), \; k \in \mathbb{Z}. \tag{3.15}$$

But the numerator is also zero at $z = i\pi/2$, therefore we suspect that $z = i\pi/2$ is a removable singularity, while the remaining values of z given in (3.15), that is $k \in \mathbb{Z}$, $k \neq -1$, are simple poles. We now verify using L'Hopital's rule in both cases:

$$\lim_{z \to i\frac{\pi}{2}} f(z) = \lim_{z \to i\frac{\pi}{2}} \frac{2i}{-i\sin(iz)} = 2,$$

and, for $k \neq -1$ we have

$$\begin{aligned}\lim_{z \to -i\left(\frac{\pi}{2}+k\pi\right)} \left(z + i\left(\frac{\pi}{2} + k\pi\right)\right) f(z) &= \lim_{z \to -i\left(\frac{\pi}{2}+k\pi\right)} \frac{(2iz - \pi) + \left(z + i\left(\frac{\pi}{2} + k\pi\right)\right) 2i}{-i\sin(iz)} \\ &= \frac{\pi + 2k\pi + 0}{i(-1)^k} \neq 0.\end{aligned}$$

The singularity at $z = i\pi/2$ may be removed by redefining f as

$$f(z) = \begin{cases} \dfrac{2iz + \pi}{\cos(iz)} & \text{if } z \neq i\dfrac{\pi}{2}, \\ 2 & \text{if } z = i\dfrac{\pi}{2}. \end{cases}$$

This redefined f is analytic at $z = i\pi/2$, but still has simple poles at $z = -i\left(\frac{\pi}{2} + k\pi\right)$, $k \in \mathbb{Z}$, $k \neq -1$. Figure 3.4 shows a phase plot where the simples poles are evident, but clearly there is no pole at $z = i\pi/2$.

Essential Singularities

Our definition (prior to learning about Laurent series) for the third type of isolated singularity is simply one that is not one of the other two types.

Definition 3.31 Isolated singularities of f that are not poles or removable singularities are **essential singularities**.

An equivalent, but better definition is given in Section 6.5, where we also discuss the behavior of functions near an essential singularity.

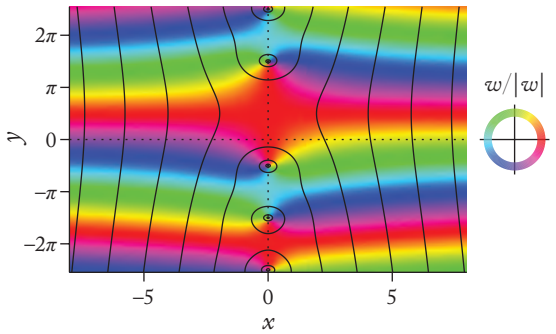


Figure 3.4: Phase plot for $f(z) = \dfrac{2iz + \pi}{\cos(iz)}$.

Example 3.32 The function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$. That $z = 0$ is not removable or a pole of any order m follows from the fact that

$$\lim_{z \to 0} z^m f(z) = \infty, \qquad \forall m \geq 0.$$

Example 3.33 The function $f(z) = e^z$ has an essential singularity at $z = \infty$ because $g(w) = f(1/w) = e^{1/w}$ has an essential singularity at $w = 0$.

3.5.2 BRANCH POINTS

The most important type of non-isolated singularity is a branch point. Branch points are singularities that occur for multivalued functions. In order to discuss branch points we will use two concepts from Section 4.1: simple closed contours and parameterizations. You may wish to read that section now, however, for the purposes of describing branch points it is enough to think of a simple closed contour as a small circle and its parameterization as a way of labeling the points on the circle with a real variable t that increases from a to b as the circle is traversed once around (for example, t could be the angle running 0 to 2π).

Let us consider the simplest multivalued function, $f(z) = z^{1/2} = \pm\sqrt{r}e^{i\theta/2}$. Consider a circle radius R centered at 0. A parameterization for this circle is $z(t) = Re^{it}$, with $0 \leq t \leq 2\pi$. Now, starting at $t = 0$, select one of the values of $f(z)$, say the one with the positive sign, traverse once around the circle (increase t from 0 to 2π), and follow the value of $f(z)$ as t increases. At the start, the value was $f(Re^{i0}) = \sqrt{R}\,e^{i0/2} = \sqrt{R}$, while at the end, the value is $f(Re^{i2\pi}) = \sqrt{R}\,e^{i2\pi/2} = \sqrt{R}e^{i\pi} = -\sqrt{R}$, which is clearly the negative of the start value. Even though the function was changing continuously as we traversed the circle, we have ended

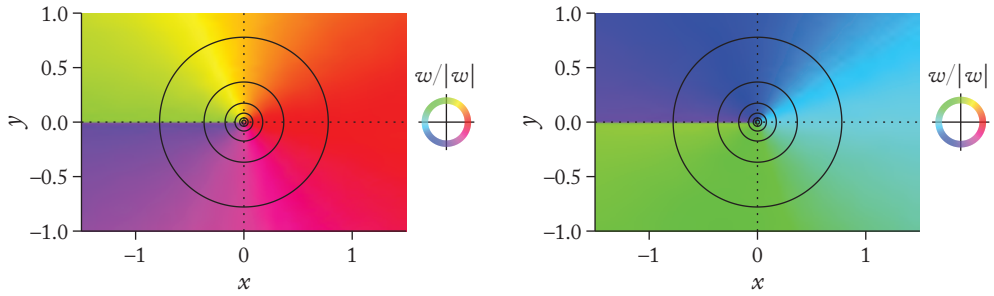


Figure 3.5: The branches $f_a(z) = \sqrt{r}\, e^{i\theta/2}$, $-\pi < \theta \le \pi$ (left), and $f_b(z) = -\sqrt{r}\, e^{i\theta/2}$, $-\pi < \theta \le \pi$ (right).

up at a different value than we had at the start. Indeed, we find that by going once around the circle we have ended up on the other branch of the square root function. Clearly, this will happen regardless of the size of the circle, provided 0 is inside the circle. Conversely, for any circle that does not contain 0 in its interior, traversing it once around will not cause the argument of z to increase by 2π and hence will not cause a change in the value of the function. Thus, the point 0 is special for this function f. Such a point is called a branch point, which we now define.

Definition 3.34 Let $f(z)$ be a multivalued function defined in a neighborhood of z_0. Let C be a simple closed contour inside this neighborhood with z_0 in its interior, and let $z(t)$, $t \in [a, b]$ be a parameterization of C (starting from some arbitrary point, going once around). Choose a value of $f(z(a))$ and allow $f(z(t))$ to vary continuously as t increases. If $f(z(a)) \neq f(z(b))$ for every such contour C, then z_0 is a **branch point** of f.

The above definition is a little technical. Basically it simply says that going once around a branch point causes the function to change value, that is, it causes a switch from one branch of the function to another. The requirement that this is true for "every such contour" means that it must be true for an arbitrarily small closed contour around z_0, thus distinguishing the point z_0. Finally, the requirement that $f(z(t))$ varies continuously along the contour prevents you from arbitrarily switching to another branch of the function at any point.

Phase plots of the two branches $f_a(z) = \sqrt{r}\, e^{i\theta/2}$, $-\pi < \theta \le \pi$, and $f_b(z) = -\sqrt{r}\, e^{i\theta/2}$, $-\pi < \theta \le \pi$ of $f(z) = z^{1/2}$ are shown in Fig. 3.5. In the process described above, we started on the positive real axis of the branch f_a, where the color is red and started moving in a circle around 0 in the counter-clockwise direction. The continuous increase of t corresponds to a continual change in the color. When the negative real axis is encountered, the value, since it must be continuous flows onto the green area of the other branch f_b. Similarly, if we were to circle the

origin again, the purple area just above the negative real axis in f_b would flow to the purple area below the negative real axis in f_a.

The function $f(z) = z^{1/2}$ also has a branch point at infinity. To show this we consider $g(w) = f(1/w)$ and consider a small circle around $w = 0$. At the start the value is $g(Re^{i0}) = (1/\sqrt{R})e^{-i0/2} = 1/\sqrt{R}$ while at the end the value is $g(Re^{i2\pi}) = (1/\sqrt{R})e^{-i2\pi/2} = -1/\sqrt{R}$. We conclude that circling 0 causes g to change branches, thus circling ∞ causes f to change branches. Therefore $f(z) = z^{1/2}$ has a branch point at infinity. It is not difficult to show that $z = 0$ and $z = \infty$ are branch points for any rational root function $f(z) = z^{p/q}$.

Example 3.35 Show that the function $\log z$ has branch points at 0 and ∞.

To show that $z = 0$ is a branch point consider a circle radius R centered at 0 and go once around. At the beginning point, select the principal value of $\log z$. The values at start and end are

$$\log(Re^{i0}) = \ln R + i0 \qquad \text{and} \qquad \log(Re^{i2\pi}) = \ln R + i2\pi.$$

Clearly these differ, for any value of $R > 0$, hence $z = 0$ is a branch point. For $z = \infty$ consider $g(w) = \log(1/w)$ and determine the values of g as you traverse a small circle around the origin, with $w = Re^{it}$, $0 \le t \le 2\pi$. The values at the start and end are

$$g(Re^{i0}) = \ln R^{-1} - i0 \qquad \text{and} \qquad g(Re^{i2\pi}) = \ln R^{-1} - i2\pi.$$

These also are not the same hence $z = \infty$ is a branch point.

The only multivalued functions we will encounter in this course are those made up of rational roots, $w^{p/q}$ (p and q are integers with no common divisor) and the logarithmic function, $\log w$. As we have seen above, the branch points of such functions occur when the input to them, w, is either 0 or ∞. So, for example, the function $f(z) = (z - i)^{2/3}$ has branch points at $z = i$ and $z = \infty$. And the function $f(z) = \log\left(\frac{1}{z^2-9}\right)$ has branch points at $z = \pm 3$ and $z = \infty$. The point $z = \infty$ is not always a branch point of such functions as the next example illustrates.

Example 3.36 Find the branch points of $f(z) = \log\left(\dfrac{z+1}{z-1}\right)$.

Let $\zeta = (z + 1)/(z - 1)$. Clearly $\zeta = 0$ when $z = -1$, hence it is a branch point. The modulus of ζ becomes unbounded only in the limit as z approaches $+1$, thus $z = +1$ is the only other branch point.

Although we have already discussed branches of multivalued functions in Section 2.2, we can now give a precise definition.

Definition 3.37 A **branch** of a multivalued function f is any single-valued function F that is analytic on some domain D and, for each $z \in D$, $F(z)$ is one of the values of $f(z)$.

The requirement for F to be analytic prevents F from arbitrarily taking on various values of f. Branches are constructed from a multivalued function by means of branch cuts.

Definition 3.38 A **branch cut** is a simple contour connecting branch points that is introduced in order to define a branch F of a multivalued function f.

(A simple contour is a curve that does not intersect itself; see Section 4.1.) The location of branch cuts is somewhat arbitrary, however they must obey the following rules.

Branch cut rules:

1. Branch cuts do not intersect themselves or each other (except at their ends).
2. Both ends of a branch cut must terminate at a distinct branch point (or at the edge of the domain on which the function is defined).
3. All branch points must be the terminal point of one or more branch cuts.

It should be noted that points on a branch cut are *not* branch points, although the branch cuts terminate at branch points.

Example 3.39 All of the following are valid branches of $f(z) = z^{1/2} = \sqrt{r}\, e^{i\frac{\theta+2\pi k}{2}}$, $k = 0, 1$ defined on the given set D. The branch cut is specified in each case and extends from the branch point at 0 to the branch point at infinity.

$$F_1(z) = \sqrt{r}\, e^{i\frac{\theta}{2}} \qquad D = \{z : -\pi < \theta \le \pi\}, \qquad \text{branch cut: } \arg(z) = \pi$$

$$F_2(z) = -\sqrt{r}\, e^{i\frac{\theta}{2}} \qquad D = \{z : 0 \le \theta < 2\pi\}, \qquad \text{branch cut: } \arg(z) = 0$$

$$F_3(z) = \sqrt{r}\, e^{i\frac{\theta}{2}} \qquad D = \left\{z : -\frac{\pi}{2} < \theta \le \frac{3\pi}{2}\right\}, \qquad \text{branch cut: } \arg(z) = \frac{3\pi}{2}$$

$$F_4(z) = -\sqrt{r}\, e^{i\frac{\theta}{2}} \qquad D = \left\{z : -\frac{4\pi}{3} < \theta \le \frac{2\pi}{3}\right\}, \qquad \text{branch cut: } \arg(z) = \frac{2\pi}{3}.$$

In the above example, the sets D are not open since they include the points on the branch cut, which are boundary points for D. However, we could make D a domain (open and connected set) by changing the loose inequality in its definition to a strict inequality, that is, by removing

the branch cut from D. We can then apply Definition 3.37 to see that each F_i is indeed analytic on a domain D and so is a branch of f.

The branch cut could be a ray at any angle θ_0, like the branch cuts in the example, but that is not necessary; it could be any curve extending from 0 to ∞ that does not intersect itself. Since the function $f(z) = z^{1/2}$ has two values for each nonzero $z \in \mathbb{C}$, we say that f has *two* branches, even though we can define an infinite number of branches by altering the branch cut. Two branches are sufficient to cover all values of $f(z) = z^{1/2}$. For example, for any fixed θ_0, all values of $f(z) = z^{1/2}$ are captured by the two branches

$$f_a(z) = \sqrt{r}\, e^{i\frac{\theta}{2}} \qquad \theta_0 \le \theta < \theta_0 + 2\pi,$$

$$f_b(z) = -\sqrt{r}\, e^{i\frac{\theta}{2}} \qquad \theta_0 \le \theta < \theta_0 + 2\pi.$$

All points on the branch cut are non-isolated singularities of the branch F, since F is discontinuous as the branch cut is crossed. Since every branch point must have at least one branch cut emanating from it, it follows that branch points are also non-isolated singularities. Although points on a branch cut are singularities of a branch F, they are *not* singularities of the multivalued function f defined on an appropriate Riemann surface, which we will discuss in Section 3.6.

Another way of defining a branch point for a multivalued function is a point where the function becomes "single-valued" in the extended complex plane. For example, $z^{p/q}$ is always 0 at $z = 0$ and always ∞ at $z = \infty$, regardless of which branch is chosen. It is also true that $\lim_{z\to 0} |\log z| = \infty$ and $\lim_{z\to\infty} |\log z| = \infty$ regardless of the branch. Thus, we see that 0 and ∞ are points where the multivalued functions $z^{p/q}$ and $\log z$ are single-valued, and hence they are branch points for these functions.

Branch cut Rule 2 indicates that a function defined on the extended complex plane cannot have exactly one branch point; if it has one then it must have at least two. This can be seen to be necessary by considering stereographic projection. An arbitrarily small circle around some point z_0 is stereographically projected to a small circle on the Riemann sphere centered at A. If z_0 is a branch point, then the value of f changes after one circuit of this circle, in other words traversing the circle causes a change in the branch of the function. The circle on the sphere can be grown, pushed over the girth of the sphere, and shrunk to an arbitrarily small circle around some other point B on the sphere. One of two things will happen. Either f continues to change value as the circle is traversed once, which means, by definition, the point that stereographically projects to B will be a branch point, or the function will cease to change value at some stage as the circle is moved over the sphere. But the latter situation indicates that the value of the function on the second branch has now become the same as the value on the starting branch, that is, the multivalued function has become single-valued at some point on the circle, which means there must be a branch point there.

Returning to the question of invertibility, recall that Theorem 3.20 indicated that an analytic function f is locally invertible at any ordinary point that is not a critical point. Why is

f not invertible at a critical point? Consider the function $f(z) = z^2 + 2z - 3$. The derivative is $f'(z) = 2z + 2$, which is zero at $z = -1$ and the image of that point is $f(-1) = -4$. To find the inverse of f solve

$$w = z^2 + 2z - 3 = (z+1)^2 - 4 \quad \Longrightarrow \quad z = (w+4)^{1/2} - 1.$$

Therefore, to obtain an analytic inverse, any one branch of $g(w) = (w+4)^{1/2} - 1$ must be selected. But g has a branch point at $w = -4 = f(-1)$, therefore g cannot be analytic in a neighborhood $w = -4$.

3.5.3 OTHER SINGULARITIES

The singularities defined in the above sections are the most important types; however, there are also other types of non-isolated singularities. For example, the function

$$f(z) = \begin{cases} z & \text{if } \mathrm{Re}(z) \geq 0, \\ z+1 & \text{otherwise.} \end{cases}$$

is analytic on the regions $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(z) < 0$, but the imaginary axis is a set of non-isolated singularities for f since the function is discontinuous there. As another example, consider the function

$$f(z) = \begin{cases} z & \text{if } \mathrm{Im}(z) \geq 0, \\ \overline{z} & \text{otherwise.} \end{cases}$$

This function is continuous everywhere, however, since we know that conjugation is not a differentiable operation, f is only analytic on the region $\mathrm{Im}(z) > 0$. We conclude that the real axis is a set of non-isolated singularities of f.

The function $f(z) = 1/\sin(1/z)$ has what is called a **non-isolated essential singularity** at $z = 0$. The fact that it is non-isolated can be seen by the fact that the points $z = 1/(k\pi)$ are all isolated singular points for f, for any nonzero $k \in \mathbb{Z}$. Thus, there are an infinite number of isolated singularities within any deleted neighborhood of 0. We will not consider such singularities in this text.

3.5.4 EXERCISES

3.15. Let m be an integer greater than one.

(a) Show that $f(z) = 1/\sin z$ has simple poles at $z = k\pi$, $k \in \mathbb{Z}$.

(b) Show that $f(z) = 1/(\sin z)^m$ has poles of order m at $z = k\pi$, $k \in \mathbb{Z}$.

(c) Is it true that if any analytic function $f(z)$ has a simple pole at z_0 then $g(z) = (f(z))^m$ has a pole of order m at z_0?

3.16. Find all singularities of $f(z) = \dfrac{1}{z\,(e^z - 1)}$ and show that they are all poles. Determine the order of these poles.

3.17. For each of the following functions, locate all singularities on the extended complex plane and name them.

(a) $\dfrac{z^2 - 4}{(z^2 + z - 2)^3}$, **(b)** $\cot(1/z)$, **(c)** $\dfrac{1}{\mathrm{Log}(iz) - \pi}$, **(d)** $\dfrac{\cos(z) - 1}{z}$, **(e)** $\dfrac{1}{\sqrt{z} - 2}$.

In part (e), $\sqrt{z}$ indicates the principal branch of the square root.

3.18. Find all branch points on the extended complex plane for the function $f(z) = \log\left((z^2 + 1)^{1/2} - i\right)$.

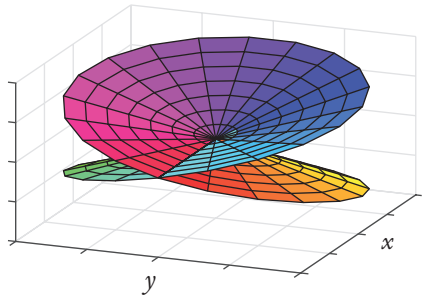
3.6 RIEMANN SURFACES

Dividing a multivalued function f into branches of single-valued functions is one way to deal with the multivaluedness. Essentially, what one is doing is making multiple copies of the codomain (the output set) of f and using each copy to accept the range of one of the branches of f. Riemann surfaces are another way of dealing with multivalued functions. In this case one makes multiple copies of the domain of f (the input set), stitched together in an appropriate manner, and considers f not as a mapping defined on $\mathbb{C}$ but rather as a mapping defined on this stitched up surface, called a **Riemann surface**. Each copy or "sheet" of this surface is mapped by a different branch of f to $\mathbb{C}$.

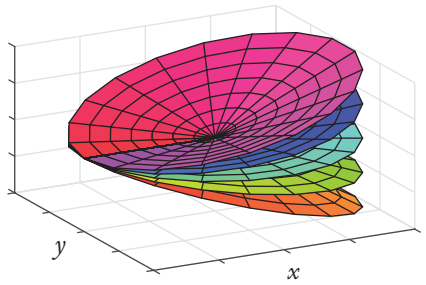
Consider again the function $f(z) = z^{1/2}$. This function has two branches $f_1(z) = \sqrt{r}\, e^{i\theta/2}$, and $f_2(z) = \sqrt{r}\, e^{i(\theta+2\pi)/2}$, where $-\pi < \theta \leq \pi$. Here, by specifying the range of θ we have indicated that the branch cut that we have chosen is the negative real axis. The Riemann surface for f is constructed by taking two copies of the complex plane and cutting each of them along this branch cut, then stitching the upper edge ($\theta = \pi$) of this cut on the first copy with the lower edge ($\theta = -\pi$) of the cut on the second copy, and vice versa. This is illustrated in Fig. 3.6a. On the first sheet of this Riemann surface, the value of the function f is $f_1(z)$, and on the second sheet it is $f_2(z)$. The function remains continuous and analytic as one travels across the stitched edges from one sheet to the other. For example, as θ approaches π on the first sheet, the value of f will approach $f_1(re^{i\pi}) = \sqrt{r}e^{i\pi/2}$. As one increases the angle further you cross the stitched edges and are on the second sheet with θ close to $-\pi$. Here the value of the function f is close to $f_2(re^{-i\pi}) = \sqrt{r}\, e^{i(-\pi+2\pi)/2} = \sqrt{r}\, e^{i\pi/2}$. If we denote the Riemann surface as S then f is a single-valued function from S to $\mathbb{C}$. It is differentiable everywhere except at the branch points $z = 0$ and $z = \infty$.

The Riemann surface for the logarithmic function has infinitely many sheets, since $\log z$ has infinitely many branches. Some of this surface is illustrated in Fig. 3.6b.

Since there is some arbitrary choice for the branch cuts, Riemann surfaces are not unique.



(a) Riemann surface for $f(z) = z^{1/2}$ with branch cut on the negative real axis



(b) Riemann surface for $f(z) = \log z$ with branch cut on the negative real axis. Five of the infinitely many sheets are shown.

Figure 3.6: Riemann surfaces for $f(z) = z^{1/2}$ and $f(z) = \log z$.

Example 3.40 Find the branch points of the function $f(z) = (z^2 - 1)^{1/2}$, and discuss the Riemann surface.

Since the function may be written as $w = f(z) = (z-1)^{1/2}(z+1)^{1/2}$, it follows that

$$\arg(w) = \frac{\arg(z-1)}{2} + \frac{\arg(z+1)}{2}.$$

As a simple closed contour is traversed once, the change in $\arg(w)$ will be half the change in $\arg(z-1)$ plus half the change in $\arg(z-1)$. For a simple closed contour that has $z = 1$ in its interior but not $z = -1$, like the dashed contour in Fig. 3.7, we see that the change in $\arg(z-1)$ will be 2π but the change in $\arg(z+1)$ will be zero. Thus, $\arg(w)$ will change by π. Similarly for a simple closed contour that has $z = -1$ in the interior but not $z = 1$, $\arg(w)$ will change by π. However, if the simple closed contour encloses both $z = 1$ and $z = -1$ then the change in $\arg(w)$ will be $\pi + \pi = 2\pi$ (which is equivalent to 0). Simple closed contours which enclose neither $z = 1$ nor $z = -1$ clearly cause no change in $\arg(w)$ as they are traversed once. We conclude that the only two branch points are $z = 1$ and $z = -1$. Exercise: Show that ∞ is not a branch point of f by considering $g(\zeta) = f(1/\zeta)$ and a small circle around $\zeta = 0$.

A branch cut for f must connect the two branch points, an obvious choice would be to take the line segment $z(t) = -1 + 2t$, $t \in [0, 1]$, but one could also take the entire real axis except that segment. In the latter case the branch cut extends from $z = -1$ decreasing along the real axis through the point at infinity and back along the positive real axis to $z = 1$. Both Riemann surfaces are shown in Fig. 3.8.

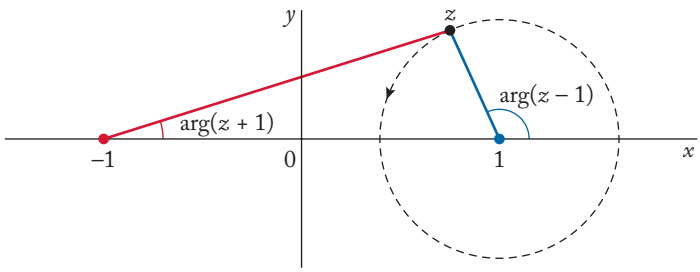


Figure 3.7: Simple closed contour around the point $z = 1$ for the function $w = (z-1)^{1/2}(z+1)^{1/2}$. Traversing the contour once, the argument of $z+1$ is unchanged while the argument of $z-1$ increases by 2π, thus $\arg(w)$ will change by $2\pi/2 = \pi$.

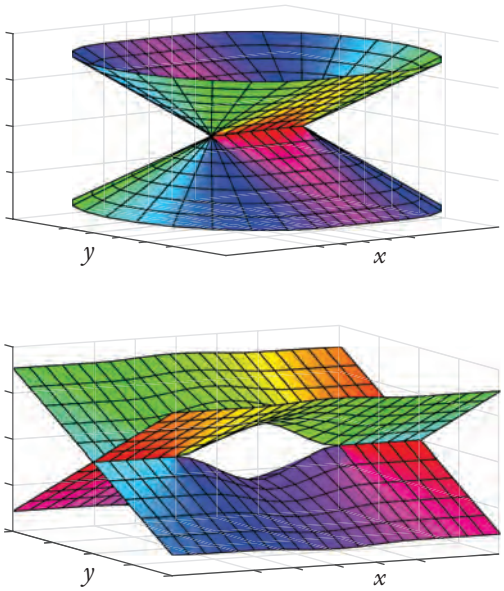


Figure 3.8: Possible Riemann surfaces for the function in Example 3.40.

CHAPTER 4

Contour Integration

4.1 ARCS, CONTOURS, AND PARAMETERIZATIONS

Definition 4.1 If $x(t)$ and $y(t)$ are continuous real-valued functions of a real parameter t, $t \in [a, b]$, $a \neq b$, then the set of points $z(t) = x(t) + iy(t)$, $t \in [a, b]$ is called an **arc**, and the function $z(t)$ is called a **parameterization** of the arc.

The requirement that $a \neq b$ in the definition is so that an arc is more than just one point.

As we have already mentioned in Section 2.3, a line through the point $B \in \mathbb{C}$ with direction $D \in \mathbb{C}$ may be written parametrically as

$$z(t) = B + Dt, \qquad t \in \mathbb{R},$$

and a circle centered at $B \in \mathbb{C}$ with radius $R \in \mathbb{R}$ may be represented parametrically as

$$z(t) = B + Re^{it}, \qquad t \in [0, 2\pi].$$

Example 4.2 Find a parameterization for the line segment joining $p_1 = -1 + i$ to $p_2 = 1 - i$.

Since the direction from p_1 to p_2 is $p_2 - p_1 = 2 - 2i$ we have

$$z(t) = -1 + i + (2 - 2i)t, \qquad 0 \leq t \leq 1.$$

The above example illustrates a more general case. A parameterization for the line segment going from p_1 to p_2 is given by

$$z(t) = p_1 + (p_2 - p_1)t, \qquad 0 \leq t \leq 1.$$

It is important to realize that parameterizations are not unique. For example, $z(t) = -1 + i + (1 - i)t$, $0 \leq t \leq 2$, is another parameterization of the straight line segment in Example 4.2. Further, although t is a common choice for the parameter name, any reasonable name can be used. Sometimes we parameterize a curve using its real or imaginary part as the parameter. For example, the curve in Example 4.2 could also be parameterized as $z(x) = x - ix$, $-1 \leq x \leq 1$.

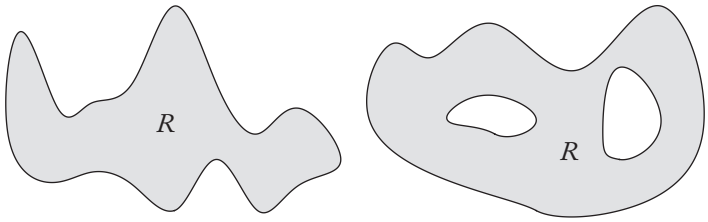


Figure 4.1: Simply (left) and multiply (right) connected sets.

Example 4.3 Find a parameterization for the half circle, radius 4 centered at $z = i$ that is in the right half-plane.

Since the half circle is to the right of the center point, a parameterization is

$$z(t) = i + 4e^{it}, \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

Definition 4.4 An arc is **simple** if it does not cross itself, that is $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$. A simple arc is also called a **Jordan arc**. If an arc is simple except for $z(a) = z(b)$ then it is called a **simple closed curve** or a **Jordan curve**.

A circle is a simple closed curve, but a figure eight is not since it crosses itself.

Recall that for R to be a connected set it means that every two points in R can be connected by a path of straight line segments completely contained in R.

Definition 4.5 A **simply connected** set is a connected set R such that every simple closed curve in R can be continuously shrunk to a point without leaving R. A connected set that is not simply connected is called **multiply connected**.

What this definition says is that a simply connected set can have no holes; see Fig. 4.1. If the set had a hole, then a simple closed curve could be drawn with the hole in its interior and it would be impossible to shrink the curve to a point without entering the hole.

4.1.1 DEFINITE INTEGRALS AND DERIVATIVES OF PARAMETERIZATIONS

Suppose that $w(t) = u(t) + iv(t)$, where u and v are piecewise continuous real functions of a real variable $t \in [a, b]$. (Piecewise continuous means that the interval $[a, b]$ can be divided into a finite number of subintervals such that w is continuous on each subinterval.) We define the

definite integral of $w(t)$ for $a \le t \le b$ as

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

Similarly, if $x(t)$ and $y(t)$ are (real) differentiable functions, then we define the derivative of $z(t) = x(t) + iy(t)$ with respect to t as

$$\frac{d}{dt}z(t) = \frac{d}{dt}x(t) + i\frac{d}{dt}y(t).$$

By these definitions it follows that

$$\operatorname{Re}\left[\int_a^b w(t)\,dt\right] = \int_a^b \operatorname{Re}[w(t)]\,dt, \qquad \operatorname{Im}\left[\int_a^b w(t)\,dt\right] = \int_a^b \operatorname{Im}[w(t)]\,dt,$$
$$\operatorname{Re}\left[\frac{d}{dt}z(t)\right] = \frac{d}{dt}\operatorname{Re}[z(t)], \qquad \operatorname{Im}\left[\frac{d}{dt}z(t)\right] = \frac{d}{dt}\operatorname{Im}[z(t)].$$

The integrals of $u(t)$ and $v(t)$ are familiar real-valued definite integrals. Thus, if $U(t)$ and $V(t)$ are (real) antiderivatives of u and v, that is $U' = u$ and $V' = v$, then

$$\int_a^b w(t)\,dt = U(b) - U(a) + i\,(V(b) - V(a)).$$

We also have the following important inequality for the modulus of this definite integral.

Theorem 4.6 *If $w(t)$, $a \le t \le b$, is a parameterization of an arc in $\mathbb{C}$ then*

$$\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt.$$

Proof. Let $\int_a^b w(t)\,dt = \rho e^{i\phi}$. Since ρ is the modulus of this integral, it is equal to the left-hand side of the inequality in the theorem statement. Then

$$\begin{aligned}\rho &= \int_a^b e^{-i\phi}w(t)\,dt = \operatorname{Re}\left[\int_a^b e^{-i\phi}w(t)\,dt\right] \qquad \text{(since } \rho \text{ is real)}\\ &= \int_a^b \operatorname{Re}\left[e^{-i\phi}w(t)\right]dt \le \int_a^b \left|e^{-i\phi}w(t)\right|dt = \int_a^b |w(t)|\,dt.\end{aligned}$$

□

Consider again the parameterization $z(t) = x(t) + iy(t)$ where x and y are differentiable functions of $t \in \mathbb{R}$. Suppose that $f = u + iv$ is an analytic function and consider $f(z(t))$. What is the derivative of this function with respect to t?

$$\begin{aligned}\frac{d}{dt} f(z(t)) &= \frac{d}{dt} u(z(t)) + i \frac{d}{dt} v(z(t)) \\ &= \lim_{\Delta t \to 0} \left(\frac{u(z(t+\Delta t)) - u(z(t))}{\Delta t} + i \frac{v(z(t+\Delta t)) - v(z(t))}{\Delta t} \right).\end{aligned}$$

Now define $\Delta z = z(t + \Delta t) - z(t)$. We may then write the above as

$$\begin{aligned}\frac{d}{dt} f(z(t)) &= \lim_{\Delta t \to 0} \left(\frac{u(z(t)+\Delta z) - u(z(t))}{\Delta z} + i \frac{v(z(t)+\Delta z) - v(z(t))}{\Delta z} \right) \left(\frac{z(t+\Delta t) - z(t)}{\Delta t} \right) \\ &= \lim_{\Delta t \to 0} \left(\frac{f(z(t)+\Delta z) - f(z(t))}{\Delta z} \right) \left(\frac{x(t+\Delta t) - x(t)}{\Delta t} + i \frac{y(t+\Delta t) - y(t)}{\Delta t} \right).\end{aligned}$$

Taking the limit as Δt goes to zero, which means that Δz also approaches zero, we immediately see that

$$\frac{d}{dt} f(z(t)) = f'(z(t)) \cdot z'(t),$$

that is, chain rule holds.

Definition 4.7 Let $z(t)$, $t \in [a, b]$ be an arc. If $z'(t)$ is continuous and nonzero on $[a, b]$ then we say the arc is **smooth**.

The unit tangent vector to a smooth arc is given by

$$T(t) = \frac{z'(t)}{|z'(t)|},$$

and the arc length is

$$L = \int_a^b \left| z'(t) \right| \, dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt. \tag{4.1}$$

4.1.2 AN APPLICATION: FOURIER SERIES

In this section we illustrate one application of integration of a complex valued function of a real variable. We shall not develop the theory of Fourier series, but take the essential points as given by the following theorem. A full exposition of real Fourier series can be found in, for example, Boyce and DiPrima [2001].

Theorem 4.8 *If a real-valued function $f(x)$ is periodic with period $2L$ and if both f and f' are piecewise continuous on the interval $[-L, L]$ then f has a Fourier series given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right), \tag{4.2}$$

where

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \quad n = 1, 2, 3, \ldots . \end{aligned} \tag{4.3}$$

The Fourier series converges to $f(x)$ at all points x where f is continuous, and converges to $(f(x+) + f(x-))/2$ at all points where f is discontinuous.

A complex form of this series can also be given. Using the definitions of cosine and sine in terms of the exponential we have

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}\right) + \frac{b_n}{2i}\left(e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{i\frac{n\pi x}{L}} + \frac{a_n + ib_n}{2} e^{-i\frac{n\pi x}{L}}. \end{aligned}$$

Define $c_0 = \dfrac{a_0}{2}$,

$$c_n = \frac{a_n - ib_n}{2}, \quad n > 0, \qquad \text{and} \qquad c_{-n} = \frac{a_n + ib_n}{2} = \overline{c_n}, \quad n > 0.$$

Then

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{i\frac{n\pi x}{L}} + \sum_{n=1}^{\infty} c_{-n} e^{-i\frac{n\pi x}{L}}.$$

The summation index in the second sum can be replaced with $-n$ and then both sums and the constant term can be combined to yield

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{i\frac{n\pi x}{L}}. \tag{4.4}$$

This is the complex form of the Fourier series for $f(x)$. The coefficients c_n for $n > 0$ are given by

$$\begin{aligned} c_n &= \frac{a_n - ib_n}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\left(\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)\right)dx \\ &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi x}{L}}\,dx. \end{aligned} \tag{4.5}$$

The above formula also holds if $n = 0$ or $n < 0$, which follows since $c_{-n} = \overline{c_n}$. In the opposite direction, we may express a_n and b_n in terms of c_n via

$$a_n = 2\,\mathrm{Re}(c_n) \qquad \text{and} \qquad b_n = -2\,\mathrm{Im}(c_n).$$

For a periodic function of period $T = 2L$ units, the frequency is $1/T$ (in cycles per unit) and the angular frequency is $\omega = 2\pi/T = \pi/L$ (in radians per unit). The Fourier series may be expressed in terms of the angular frequency as

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{in\omega x}, \quad c_n = \frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} f(x)e^{-in\omega x}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{x}{\omega}\right)e^{-inx}\,dx, \tag{4.6}$$

where to obtain the last equality we have done a change of variables $x \mapsto x/\omega$ in the integral.

The utility of the complex form of the Fourier series is apparent when one needs to compute the Fourier coefficients for a given function $f(x)$. Even for simple functions, obtaining the values a_n and b_n from (4.3) typically involves multiple integration by parts steps, whereas obtaining c_n from (4.5) typically involves half as many such steps. We illustrate with an example.

Example 4.9 Find the Fourier series for $f(x) = x$, $f(x + 2L) = f(x)$.

From (4.6) we have

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{x}{\omega}e^{-inx}\,dx = \frac{1}{2\pi}\left(\left[\left(\frac{x}{\omega}\right)\frac{-1}{in}e^{-inx}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{1}{\omega}\right)\frac{-1}{in}e^{-inx}\,dx\right) \\ &= \frac{1}{2\pi}\left(\frac{-\pi}{in\omega}\left(e^{-in\pi} + e^{in\pi}\right) - \left[\frac{1}{(in)^2\omega}e^{-inx}\right]_{-\pi}^{\pi}\right) \\ &= \frac{1}{2\pi}\left(\frac{-\pi}{in\omega}\left((-1)^n + (-1)^n\right) + \frac{1}{n^2\omega}\left(e^{-in\pi} - e^{in\pi}\right)\right) \\ &= \frac{-1}{in\omega}(-1)^n + \frac{1}{2\pi n^2\omega}\left((-1)^n - (-1)^n\right) = i\frac{(-1)^n}{n\omega}. \end{aligned}$$

Thus, the Fourier series for f is

$$f(x) = \sum_{n\in\mathbb{Z}} i\frac{(-1)^n}{n\omega}e^{in\omega x}.$$

If we wanted the series in the real form, then, since $a_n = 2\,\mathrm{Re}(c_n)$ and $b_n = -2\,\mathrm{Im}(c_n)$ it follows that $a_n = 0$ for all n and

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2L}{n\pi} \sin\left(\frac{n\pi x}{L}\right).$$

4.1.3 CONTOURS

Definition 4.10 A **contour** is a piecewise smooth arc, that is, a finite number of smooth arcs joined end to end. If this arc is simple, it is called a **simple contour**. If it is also closed, it is called a **simple closed contour**.

If $z(t)$ is a parameterization of a contour, then, since the contour is a piecewise smooth arc it follows that $z(t)$ is a piecewise differentiable function, that is, $z'(t)$ exists on each arc that composes the contour.

Example 4.11 Find a parameterization for the contour made up of the sides of the unit square with corners at 0, 1, $1 + i$, and i.

A parameterization of this contour is

$$z(t) = \begin{cases} t & \text{if } 0 \le t < 1, \\ 1 + i(t-1) & \text{if } 1 \le t < 2, \\ (3-t) + i & \text{if } 2 \le t < 3, \\ i(4-t) & \text{if } 3 \le t < 4. \end{cases}$$

In the above example, the contour is composed of four smooth arcs (in this case line segments). Rather than writing a single parameterization valid for the entire contour, as in the above example, one often parameterizes each arc separately. Thus, for example, the fourth arc in the above example could be parameterized as $z_4(t) = i(1-t)$, $0 \le t < 1$.

Contours that are *not* closed, that is, contours that have two distinct end points, can be given a direction by specifying which of the two points is the "start" point and which is the "finish" point. This specification can be made without reference to any parameterization of the contour. The positive direction is going from start to finish, while going from finish to start is called the negative direction. If C is a contour with a specified direction, then we use the notation $-C$ to represent the same contour but traversed in the negative direction. Parameterizations of contours are generally chosen so that the parameter increases as the contour is traversed in the

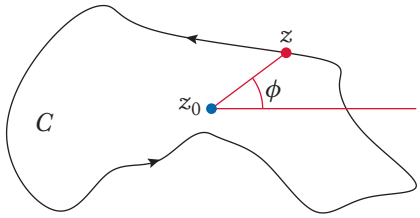


Figure 4.2: Positive direction of a simple closed contour. The angle $\phi = \arg(z - z_0)$ increases by 2π as the contour is traversed once in the positive sense.

positive sense. It is sometimes useful to note that if $z(t)$, $t \in [a, b]$ is a parameterization of a contour C, then $w(t) = z(a + b - t)$, $t \in [a, b]$ is a parameterization of $-C$.

In contrast, a simple closed contour does not have distinguished start and finish points, and so a positive direction cannot be specified in the same way as it was done for non-closed contours. If one has a specific parameterization for the closed contour, a positive direction could be defined as that associated with the increase of the parameter. However, there is a more intrinsic way of defining a positive direction for simple closed contours that does not rely on any parameterization. To do so, we need the result of the following theorem.

Theorem 4.12 The Jordan Curve Theorem. *A Jordan curve (may have infinite length) divides the plane into two regions having the curve as a common boundary. One region is bounded (called the interior) and one region is unbounded (called the exterior).*

The Jordan curve theorem is one of those theorems in mathematics that appears so obvious it shouldn't need to be a theorem, yet the proof of this theorem is not straight-forward and is beyond the scope of this text.[1] We shall take it for granted that a simple closed curve has a well-defined interior that has finite area.

With the above result, the positive direction of a simple closed contour is, by convention, defined as follows.

Definition 4.13 Let C be a simple closed contour and let z_0 be a point in the interior of C. Let z be any point on C. The positive direction of C is the direction that causes $\arg(z - z_0)$ to increase by 2π after z completes one full traversal of C.

This definition is illustrated in Fig. 4.2. Since z_0 is in the interior of C, allowing z to traverse all of C once will necessarily cause $\phi = \arg(z - z_0)$ to increase by 2π as the curve is traversed in the indicated direction. If z_0 was in the exterior of C, the argument of $z - z_0$ would be the same after one full traversal of C, regardless of direction.

[1] See Newman [1964, pp. 115–116] for a proof.

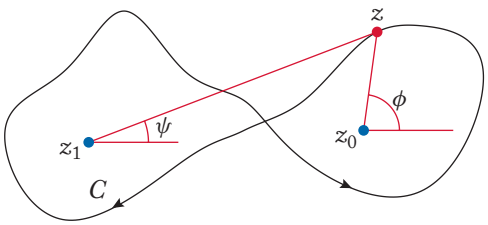


Figure 4.3: There is no preferred direction for non-simple closed contours. The angle $\phi = \arg(z - z_0)$ increases by 2π but the angle $\psi = \arg(z - z_1)$ decreases by 2π as the curve C is traversed once in the specified direction.

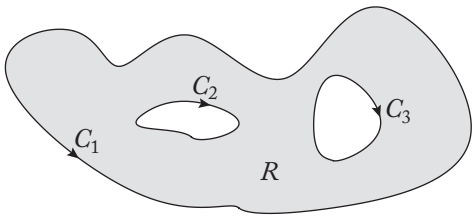


Figure 4.4: Positive direction of the boundary of a multiply connected region.

Practically speaking, the positive direction of a simple closed contour is the counter-clockwise direction, when viewing the complex plane from above. Another equivalent way of specifying this direction is to say that a person walking on the complex plane along the contour in the positive direction must keep the interior of the contour to the left.

For closed contours that are *not* simple, like a figure eight, there is no way to define a positive direction except through a specific parameterization; see Fig. 4.3. In these cases, if the direction matters one must explicitly specify the direction.

If a set of simple closed contours, C_k, $1 \le k \le n$, are the boundary of a multiply connected region R, then by convention, we define the positive orientation of the *boundary* of R by taking the orientations of the C_k as follows:

- if R is in the interior of C_k then the positive orientation of C_k is used;
- otherwise (R is not in the interior of C_k) the negative orientation of C_k is used.

See Fig. 4.4. Again, if one adheres to the rule that the interior of R is kept to the left, you will obtain the positive orientation.

4.1.4 EXERCISES

4.1. Let $w(t)$, $\pi/4 \le t \le 3\pi/4$ be the parameterization of the quarter circle centered at $z_0 = 1 - i$ and running from 2 to 0 in the upper half-plane. Compute $\int_{\pi/4}^{3\pi/4} w(t)\,dt$.

4.2. **(a)** Evaluate $\int_1^3 \left(ti - \frac{1}{t}\right)\, dt$.

(b) For $z \in \mathbb{C}$, when does $\int_0^\infty e^{-zt}\, dt$ exist, and what is its value?

4.3. Let $w(t)$, $a \le t \le b$ be a parameterization of some smooth arc Γ. Use the result of Theorem 4.6 and the definition of arc length given by (4.1) to show that the distance between the end points $w(b)$ and $w(a)$, is always less than or equal to the arc length of Γ.

4.4. In first-year calculus if asked to compute

$$A = \int_0^x e^t \cos t\, dt \quad \text{and} \quad B = \int_0^x e^t \sin t\, dt,$$

you would have done so via integration by parts. Instead, note that $A + iB = \int_0^x e^{(1+i)t}\, dt$ and hence evaluate A and B by evaluating this latter integral and then identifying the real and imaginary parts.

4.5. Find the complex form of the Fourier series for piecewise constant function

$$f(x) = \begin{cases} 1 & \text{if } -L < x \le 0, \\ 3 & \text{if } 0 < x \le L, \end{cases}$$

where f is defined elsewhere by $f(x + 2L) = f(x)$.

4.6. Find the complex form of the Fourier series for the sawtooth function

$$f(x) = \begin{cases} 1 + x & \text{if } -1 < x \le 0, \\ 1 - x & \text{if } 0 < x \le 1, \end{cases}$$

where f is defined elsewhere by $f(x + 2) = f(x)$.

4.2 CONTOUR INTEGRALS

Let C be a contour, with specified positive direction, and let $f(z)$ be a piecewise continuous function on $\mathbb{C}$. The contour integral of f along C is written as

$$\int_C f(z)\, dz.$$

If C is a simple closed contour, then we usually use the notation

$$\oint_C f(z)\, dz,$$

to emphasize that the contour is closed.

Definition 4.14 The contour integral of $f(z)$ along the contour C is defined as

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt, \tag{4.7}$$

where $z(t)$, $t \in [a,b]$ is a parameterization of the contour.

All we have done in this definition is replace z with $z(t)$ (since we must be on the contour) and replace dz with $z'(t)dt$. Equation (4.7) may also be written as

$$\int_C f(z)\,dz = \int_C (u+iv)\,(dx+i\,dy) = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy,$$

where $f = u+iv$ and $z = x+iy$. The two integrals on the right are real line integrals in $\mathbb{R}^2$. If a parameterization of C in $\mathbb{C}$ is $z(t) = x(t)+iy(t)$, $t \in [a,b]$, then the parameterization of C in $\mathbb{R}^2$ is $(x(t), y(t))$. It is easy to show that evaluating the real line integrals with this parameterization is equivalent to (4.7).

We now show that the definition given by (4.7) is reasonable, that is, does not depend on the specific parameterization chosen.

Theorem 4.15 Independence of Parameterization. *The definition of a contour integral given by* (4.7) *is independent of the parameterization chosen.*

Proof. Let $z(t), t \in [a,b]$, and $w(s), s \in [c,d]$ be two parameterizations of C. Since both parameterizations trace out the points in C it follows that we can define a piecewise smooth invertible function $t = g(s)$ via the relationship $z(t) = w(s)$ for $s \in [c,d]$. Then applying this change of variables to the right side of (4.7) we have

$$\int_a^b f(z(t))\frac{dz(t)}{dt}\,dt = \int_c^d f(z(g(s)))\left(\frac{dw(s)}{ds}\frac{ds}{dt}\right)\left(\frac{dg(s)}{ds}\,ds\right).$$

But $s = g^{-1}(t)$ so $\dfrac{ds}{dt} = \left(\dfrac{dg(s)}{ds}\right)^{-1}$, therefore,

$$\int_a^b f(z(t))\frac{dz(t)}{dt}\,dt = \int_c^d f(w(s))\frac{dw(s)}{ds}\,ds.$$

□

It is not difficult to show that contour integration also satisfies the following properties.

1. Contour integration is a linear operation.

$$\int_C Af(z) + Bg(z)\,dz = A\int_C f(z)\,dz + B\int_C g(z)\,dz,$$

for functions f and g and constants A and B.

2. Reversing the direction negates the value of the contour integral.

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz.$$

To show this we use the fact that if $z(t), t \in [a,b]$ is a parameterization of C then $w(t) = z(a+b-t), t \in [a,b]$ is a parameterization of $-C$.

$$\begin{aligned}\int_{-C} f(w)\,dw &= \int_a^b f(w(t))w'(t)\,dt = \int_{t=a}^{t=b} f(z(a+b-t)z'(a+b-t)(-1)\,dt\\ &= \int_{s=b}^{s=a} f(z(s))z'(s)\,ds, \qquad \text{where } s = a+b-t, \text{ so that } ds = -dt,\\ &= -\int_a^b f(z(s))z'(s)\,ds = -\int_C f(z)\,dz.\end{aligned}$$

3. If C can be split into two contours C_1 and C_2 with matching direction then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

Example 4.16 Evaluate $\oint_C f(z)\,dz$ where $f(z) = \overline{z}$ and C is the unit circle centered at 0.

A parameterization of C is $z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$. We have that $z'(\theta) = ie^{i\theta}$ so that

$$\oint_C f(z)\,dz = \int_0^{2\pi} \overline{e^{i\theta}}ie^{i\theta}\,d\theta = \int_0^{2\pi} e^{-i\theta}ie^{i\theta}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i.$$

Example 4.17 Repeat the previous example but with $f(z) = z^2$.

$$\begin{aligned}\oint_C f(z)\,dz &= \int_0^{2\pi} e^{i2\theta}ie^{i\theta}\,d\theta = i\int_0^{2\pi} e^{i3\theta}\,d\theta\\ &= i\left[\frac{1}{3i}e^{i3\theta}\right]_0^{2\pi} = \frac{1}{3}\left[e^{i6\pi} - e^{i0}\right] = \frac{1}{3}[1-1] = 0.\end{aligned}$$

The next example is more involved in that the region is multiply connected and the simple closed contours that make up its boundary are composed of four arcs each.

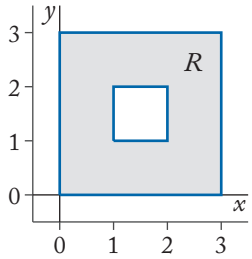


Figure 4.5: The region R for Example 4.18.

Example 4.18 Find $\oint_C f(z)\,dz$ where $f(z) = xyi$ and C is the boundary of the region R as shown in Fig. 4.5.

Parameterizations for the four arcs composing the outer boundary and the four arcs composing the inner boundary are:

outer boundary	inner boundary
$z_1(t) = t, \quad t \in [0,3];$	$z_5(t) = t + 2i, \quad t \in [1,2];$
$z_2(t) = 3 + it, \quad t \in [0,3];$	$z_6(t) = 2 + i(2-t), \quad t \in [0,1];$
$z_3(t) = 3 - t + 3i, \quad t \in [0,3];$	$z_7(t) = 2 - t + i, \quad t \in [0,1];$
$z_4(t) = i(3-t), \quad t \in [0,3];$	$z_8(t) = 1 + ti, \quad t \in [1,2].$

Therefore,

$$\begin{aligned}
\oint_C f(z)\,dz &= \int_0^3 (t)(0)i(1)\,dt + \int_0^3 (3)(t)i(i)\,dt + \int_0^3 (3-t)(3)i(-1)\,dt \\
&\quad + \int_0^3 (0)(3-t)i(-i)\,dt + \int_1^2 (t)(2)i(1)\,dt + \int_0^1 (2)(2-t)i(-i)\,dt \\
&\quad + \int_0^1 (2-t)(1)i(-1)\,dt + \int_1^2 (1)(t)i(i)\,dt \\
&= 0 + \left[-\frac{3}{2}t^2\right]_0^3 + \left[-3i\left(3t - \frac{1}{2}t^2\right)\right]_0^3 + 0 + \left[it^2\right]_1^2 + \left[4t - t^2\right]_0^1 \\
&\quad + \left[-i\left(2t - \frac{1}{2}t^2\right)\right]_0^1 + \left[-\frac{1}{2}t^2\right]_1^2 \\
&= -\frac{27}{2} - \frac{27i}{2} + 3i + 3 - \frac{3i}{2} - \frac{3}{2} = -12 - 12i.
\end{aligned}$$

The result of the following theorem will be used extensively. It simply states that the modulus of the contour integral must be below the maximum value of the modulus on the contour times the arc length of the contour.

Theorem 4.19 Contour Integral Bound. *Let C be a contour and f a piecewise continuous function defined on C. Let $M \in \mathbb{R}$ such that $|f(z)| \leq M$ for all $z \in C$, and let L be the arc length of C. Then*

$$\left| \int_C f(z)\, dz \right| \leq ML.$$

Proof. Let $z(t)$, $t \in [a, b]$ be a parameterization of C.

$$\begin{aligned} \left| \int_C f(z)\, dz \right| &= \left| \int_a^b f(z(t))z'(t)\, dt \right| \leq \int_a^b \left| f(z(t))z'(t) \right| dt, \qquad \text{by Theorem 4.6,} \\ &= \int_a^b \left| f(z(t)) \right| \left| z'(t) \right| dt \leq \int_a^b M \left| z'(t) \right| dt = M \int_a^b \left| z'(t) \right| dt = ML. \end{aligned}$$

□

4.2.1 EXERCISES

4.7. As indicated in the discussion following Definition 4.14, a contour integral may be defined as

$$\int_C f(z)\, dz = \int_C u\, dx - v\, dy + i \int_C v\, dx + u\, dy,$$

where $f = u + iv$ and the integrals on the right are real line integrals on $\mathbb{R}^2$. If $(x(t), y(t))$, $t \in [a, b]$, is a parameterization of C in $\mathbb{R}^2$, show that these line integrals evaluate to an expression equivalent to (4.7).

4.8. Let C be the straight line from $(-1 - i)$ to 1. Evaluate

(a) $\displaystyle\int_C z + \frac{1}{z}\, dz,$ **(b)** $\displaystyle\int_C \mathrm{Re}(z) + \mathrm{Im}(z)\, dz.$

4.9. Compute $\int_C \left(\overline{z}^2 - 2i\right) dz$, where C is the semi-circle of radius 3 centered at $1 - i$ starting at $z = 1 + 2i$ and passing through the region $\mathrm{Re}(z) \leq 1$.

4.10. Compute $\int_C \left(x^2 - iy\right) dz$, where $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$, and C is:

(a) the straight line from $z = 1 + 2i$ to $z = 2 + 8i$,

(b) the parabola $y = 2x^2$ from $z = 1 + 2i$ to $z = 2 + 8i$.

4.11. For each given function $f(z)$ compute the contour integral $\int_C f(z)\,dz$ for the given contour C.

(a) $f(z) = z\overline{z}$, C is the straight line from 0 to $1+i$.

(b) $f(z) = z\overline{z}$, C is the straight line from 0 to 1, followed by the straight line from 1 to $1+i$.

(c) $f(z) = \frac{1}{r} - 2i\cos(\theta)$, C is the semicircle of radius 3 in the region $y \geq 0$ centered at the origin traversed counter-clockwise. [Hint: It is helpful to express cosine in terms of exponentials.]

(d) $f(z) = z^{1/2}$, C is the quarter circle, radius 2 centered at the origin going from $\sqrt{2}(-1+i)$ to $\sqrt{2}(-1-i)$. First do this integral using the principal branch of $z^{1/2}$. Be careful since the contour crosses the branch cut; to cope with this parameterize the contour in two pieces. Does the value of the integral change if a branch is chosen so that C does not cross a branch cut?

4.12. For each of the following regions D with boundary C show that

$$\oint_C \overline{z}\,dz = 2i(\text{area of } D). \tag{4.8}$$

(a) D is the disk radius R centered at z_0.

(b) D is the rectangle with vertices $a+ib$, $(a+W)+ib$, $(a+W)+i(b+L)$, $a+i(b+L)$, where a and b are real numbers, and W and L are positive real numbers.

Do you think that the formula (4.8) is valid for any region D? If so, give a brief argument as to why you think it holds.

4.13. Let $A = 2$ and $B = -2i$. Let C_1 be the path from A to B counterclockwise along the circle or radius two centered at the origin. Let C_2 be the path from A to B along the straight line joining them. For each of the following functions f, explicitly compute the contour integral of f along C_1 by separating the integrand into real and imaginary parts and then integrating each separately. Then do the same for C_2. If the values of $\int_{C_1} f(z)\,dz$ and $\int_{C_2} f(z)\,dz$ are the same, show that f is analytic in a simply connected region enclosing the curves, if the values are different, show that f is not analytic in such a region.

(a) $f(z) = \dfrac{1}{z}$, **(b)** $f(z) = z^2$.

4.14. Using the inequality $\left|\int_C f(z)\,dz\right| \leq ML$, where L is the arclength of C, and $M = \max_{z\in C} |f(z)|$, show the following.

(a) If C is the arc of the circle $|z| = 1$ that lies in the quadrant $x \geq 0$, $y \leq 0$, then $\left| \int_C \operatorname{Log} z \, dz \right| \leq \frac{\pi^2}{4}$.

(b) If C is the circle $|z| = 3$ (traversed once), then $\left| \int_C \frac{dz}{z^2 - i} \right| \leq \frac{3\pi}{4}$.

CHAPTER 5

Cauchy Theory

In this chapter we will begin to see more manifestations of how the definition of the complex derivative imposes strong conditions on analytic functions. The central results are the Cauchy–Goursat theorem and the Cauchy integral formulas. Many results follow from these.

5.1 THE CAUCHY–GOURSAT THEOREM AND ITS CONSEQUENCES

Before we state and prove the Cauchy–Goursat theorem, we will need to recall a result from real multivariable integral calculus regarding integrals over regions in the plane and line integrals around the boundary. This result was formulated by George Green (1793–1841); a proof can be found in most multivariable calculus texts, for example, Taylor and Mann [1983, pp. 457–464]. We state his theorem without proof.

Theorem 5.1 Green's Theorem. *If $P(x, y)$ and $Q(x, y)$ are continuous and have continuous partial derivatives in a region R of the plane and on its boundary C, then*

$$\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dx dy.$$

This theorem is valid for both simply and multiply connected regions. It is a rather remarkable result. As long as the functions P and Q have continuous partial derivatives, Green's theorem tells us that a certain difference of these partial derivatives when integrated over a region is the same as a certain line integral of the functions themselves on the boundary of the region. Why information about a function in the interior of a region should dictate information about the function on the boundary (and vice versa) is somewhat surprising.

We now state the central result. Recall that by definition a "region" is connected.

Theorem 5.2 Cauchy–Goursat Theorem. *If $f(z)$ is an analytic in a region R and on its boundary C then*

$$\oint_C f(z)\,dz = 0.$$

Cauchy first proved this theorem in 1814 with the added restriction that $f'(z)$ is continuous. His proof used a formula equivalent to Green's theorem, which itself was not formulated until 1828. Our proof below will use Green's theorem and have the added restriction. Édouard Goursat (1858–1936) later proved the theorem without this added restriction.[1] This is an important generalization because, as we shall see later, it will eventually exclude the possibility of there being an analytic function whose derivative is not continuous.

Proof. We will assume, as in Cauchy's proof, that $f'(z)$ is continuous in R and on its boundary. Let $z = x + iy$ and $f = u + iv$ where u and v are real functions of x and y. Since f' is continuous, it follows that u and v have continuous partial derivatives. Therefore,

$$\begin{aligned}\oint_C f(z)\,dz &= \oint_C (u+iv)(dx+i\,dy) = \oint_C u\,dx - v\,dy + i\oint_C v\,dx + u\,dy \\ &= \iint_R (-v_x - u_y)\,dx\,dy + i\iint_R (u_x - v_y)\,dx\,dy \qquad \text{(by Green's theorem)}, \\ &= \iint_R 0\,dx\,dy + i\iint_R 0\,dx\,dy \qquad \text{(by the CR equations)}, \\ &= 0.\end{aligned}$$

An equivalent way of stating the Cauchy–Goursat theorem is to say that if f is analytic in a region R then $\oint_C f(z)\,dz = 0$ for all closed contours C contained in R. □

Example 5.3 Let C be the unit circle centered at the origin. Explicitly compute the contour integral of $f(z) = z^2$ around C to verify the Cauchy–Goursat theorem in this case.

A parameterization of C is $z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$. Thus,

$$\int_C f(z)\,dz = \int_0^{2\pi} \left(e^{i\theta}\right)^2 i e^{i\theta}\,d\theta = \int_0^{2\pi} i e^{i3\theta}\,d\theta = \left[\frac{e^{i3\theta}}{3}\right]_0^{2\pi} = \frac{1}{3} - \frac{1}{3} = 0.$$

The Cauchy–Goursat theorem is quite a strong statement. No matter what function $f(z)$ you have, as long as it is differentiable everywhere in R, the integral of f around the boundary will be zero. Since Green's theorem holds for multiply connected regions, so does the Cauchy–Goursat theorem. There are a number of results that are consequences of the Cauchy–Goursat theorem, which we now state and prove.

5.1.1 PATH INDEPENDENCE

The first consequence of the Cauchy–Goursat theorem is that for analytic functions which path you take from p_1 to p_2 does not matter; this is called path independence.

[1] For a proof of the theorem without the requirement that $f'(z)$ be continuous, see Churchill and Brown [1990, pp. 109–113] or Zill and Shanahan [2015, App. II].

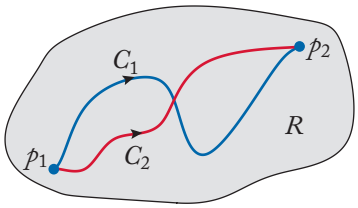


Figure 5.1: Path independence.

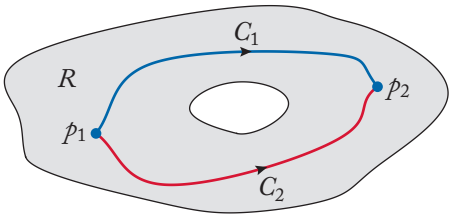


Figure 5.2: Path independence does not hold for multiply connected regions.

Theorem 5.4 Path Independence. *Let $f(z)$ be an analytic function in a simply connected region R. Let p_1 and p_2 be two points in R, and let C be any contour in R from p_1 to p_2. Then $\int_C f(z)\,dz$ is independent of the contour chosen, that is, its value depends only on p_1 and p_2.*

Proof. Let C_1 and C_2 be any two contours in R from p_1 to p_2 as illustrated in Fig. 5.1. Then $C_1 - C_2$ forms a (possibly non-simple) contour whose interior D is contained in R. We may therefore apply the Cauchy–Goursat theorem to $C_1 - C_2$ yielding

$$0 = \oint_{C_1 - C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz \quad \Longrightarrow \quad \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

□

This theorem does not apply if R is multiply connected since in this case you may get a situation as displayed in Fig. 5.2 where the region D interior to $C_1 - C_2$ is not contained in R. One should also note that technically one does not need f to be analytic to get the same result; the only thing that is needed is the *result* of the Cauchy–Goursat theorem, namely that the integral around all closed contours is zero.

Example 5.5 Let $f(z) = \cos z$, $p_1 = -i\pi$, and $p_2 = i\pi$. Let C_1 be the straight line $z_1(t) = -i\pi + i2\pi t$, $0 \le t \le 1$, and C_2 be the semicircle $z_2(t) = \pi e^{it}$, $-\pi/2 \le t \le \pi/2$. Compute the contour integrals of f along C_1 and C_2 to verify that they are the same.

We have for C_1,

$$\int_{C_1} f(z)\,dz = \int_0^1 \cos(-i\pi + i2\pi t)2\pi\,dt = [\sin(-i\pi + i2\pi t)]_0^1 = \sin(i\pi) - \sin(-i\pi)$$
$$= i\sinh\pi + i\sinh\pi = 2i\sinh\pi.$$

While for C_2, we have

$$\int_{C_2} f(z)\,dz = \int_{-\pi/2}^{\pi/2} \cos\left(\pi e^{it}\right) i\pi e^{it}\,dt = \left[\sin\left(\pi e^{it}\right)\right]_{-\pi/2}^{\pi/2} = \sin(i\pi) - \sin(-i\pi)$$
$$= i\sinh\pi + i\sinh\pi = 2i\sinh\pi.$$

5.1.2 COMPLEX EXTENSION OF THE FUNDAMENTAL THEOREM OF CALCULUS

Path independence allows us to define unambiguously the function

$$F(z) = \int_A^z f(w)\,dw, \tag{5.1}$$

where A and z are both in a simply connected region R, and f is analytic in R. The path is not specified in (5.1), but it does not matter; the path independence theorem allows us to take any path from A to z in R, and tells us that it will always evaluate to the same value. Using this definition of F we can now state the second consequence of the Cauchy–Goursat theorem.

Theorem 5.6 Complex Extension of the Fundamental Theorem of Calculus. *Let f be analytic in a simply connected region R, then for $A, z \in R$, the function defined by (5.1) is an antiderivative of f in R, that is*

$$\frac{d}{dz}\int_A^z f(w)\,dw = f(z). \tag{5.2}$$

Furthermore, for any contour C joining p_1 to p_2 in R we have

$$\int_C f(z)\,dz = \int_{p_1}^{p_2} f(z)\,dz = F(p_2) - F(p_1). \tag{5.3}$$

Proof.

$$F(z+\Delta z) - F(z) = \int_A^{z+\Delta z} f(w)\,dw - \int_A^z f(w)\,dw = \int_z^{z+\Delta z} f(w)\,dw.$$

For $|\Delta z|$ sufficiently small we may take the path from z to $z + \Delta z$ as the straight line segment $z + t\Delta z$, $0 \le t \le 1$. Thus,

$$F(z + \Delta z) - F(z) = \int_0^1 f(z + t\Delta z)\Delta z\, dt = \Delta z \int_0^1 f(z + t\Delta z)\, dt$$

so that

$$\lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = \lim_{\Delta z \to 0} \int_0^1 f(z + t\Delta z)\, dt = f(z) \int_0^t dt = f(z).$$

By the definition of the derivative, we have established (5.2), namely that $F'(z) = f(z)$ for all $z \in R$.

To establish the second part of the theorem we first note that since f is analytic, the path independence theorem holds and so the first equality sign in (5.3) holds. To show the second equality sign we add to the contour C a contour from A to p_1 inside R, and again, since path independence holds, the exact contour does not matter. Then

$$\begin{aligned}\int_C f(z)\, dz = \int_{p_1}^{p_2} f(z)\, dz &= -\int_A^{p_1} f(z)\, dz + \int_A^{p_1} f(z)\, dz + \int_{p_1}^{p_2} f(z)\, dz \\ &= -\int_A^{p_1} f(z)\, dz + \int_A^{p_2} f(z)\, dz = -F(p_1) + F(p_2).\end{aligned}$$

□

As with the real case, complex antiderivatives are not unique; an arbitrary constant can be added to them since if $F'(z) = f(z)$ and $K \in \mathbb{C}$ is a constant, then $G(z) = F(z) + K$ also gives $G'(z) = f(z)$. And conversely, if F and G are antiderivatives of f in R then $(F - G)' = 0$ implying $F - G$ is a constant in R. We may thus define indefinite integrals in R when f is analytic via

$$\int f(z)\, dz = F(z) + K, \qquad K \in \mathbb{C},$$

where F is an antiderivative of f in R.

Example 5.7 Find antiderivatives of the following functions: (a) $\sin(3z)\cos(3z)$, (b) $\cot(4z - i)$, and (c) ze^{2z}.

(a) $$\int \sin(3z)\cos(3z)\, dz = \frac{1}{6}\sin^2(3z) + K,$$

(b) $$\int \cot(4z - i)\, dz = \int \frac{\cos(4z - i)}{\sin(4z - i)}\, dz = \frac{1}{4}\log\left(\sin(4z - i)\right) + K,$$

(c) $$\int ze^{2z}\, dz = \frac{z}{2}e^{2z} - \int \frac{1}{2}e^{2z}\, dz = \frac{z}{2}e^{2z} - \frac{1}{4}e^{2z} + K. \qquad \text{(integration by parts)}$$

The next two examples illustrate how to deal with situations when the antiderivative is a multivalued function. Essentially, one must choose an appropriate branch of the function.

Example 5.8 Consider the branch of $z^{1/2}$ defined by $z^{1/2} = \sqrt{r}e^{i\theta/2}$, $-3\pi/2 \le \theta < \pi/2$. (This function has a branch point at $z = 0$ and a branch cut on the positive imaginary axis.) Evaluate $\int_C z^{1/2}\,dz$, where C is any contour joining $p_1 = -1 + i\sqrt{3}$ to $p_2 = 3$ such that the contour does not touch the nonnegative imaginary axis.

An antiderivative of $z^{1/2}$ is $\frac{2}{3}z^{3/2}$, which itself is a multivalued function. In this case we must choose the branch cut for the antiderivative in the same location as for the integrand and we must choose the branch of $z^{3/2}$ so that its derivative is the integrand. So we must choose

$$z^{3/2} = r^{3/2}e^{i3\theta/2}, \qquad -\frac{3\pi}{2} \le \theta < \frac{\pi}{2}.$$

We could not choose the branch

$$z^{3/2} = r^{3/2}e^{i3(\theta+2\pi)/2}, \qquad -\frac{3\pi}{2} \le \theta < \frac{\pi}{2},$$

even though it is a valid branch of $z^{3/2}$, because its derivative is the opposite branch of $z^{1/2}$ than the one that the problem specified. Therefore, using the appropriate branch of $z^{3/2}$ we have

$$\begin{aligned}\int_C z^{1/2}\,dz &= \left[\frac{2}{3}z^{3/2}\right]_{-1+i\sqrt{3}}^{3} = \frac{2}{3}\left[3^{3/2} - (-1+i\sqrt{3})^{3/2}\right] \\ &= \frac{2}{3}\left[(3)^{3/2}e^{i3(0)/2} - (2)^{3/2}e^{i3(-4\pi/3)/2}\right] \\ &= \frac{2}{3}\left[3\sqrt{3} - 2\sqrt{2}e^{-i2\pi}\right] = \frac{2}{3}\left[3\sqrt{3} - 2\sqrt{2}\right].\end{aligned}$$

Note that when determining the arguments of the two end points we chose $-4\pi/3$ and 0; in particular we did not choose $2\pi/3$ as the argument for $-1 + i\sqrt{3}$. This is because it is necessary to choose both arguments in the same range of θ so that getting from one to the other does not require a crossing of the branch cut.

Example 5.9 Compute $I = \int_C \frac{3}{2z-4}\,dz$. where the contour C goes from $-2i$ to $2i$ but does not cross the positive real axis with $x \ge 2$.

Since $\log z$ is an antiderivative of $1/z$ we have that

$$I = \left[\frac{3}{2}\log(2z-4)\right]_{-2i}^{2i}.$$

However, the function $\log(2z-4)$ is multivalued. Which branch is to be chosen? It has a branch point at $z = 2$ and a branch cut from there going in some direction to infinity. Due

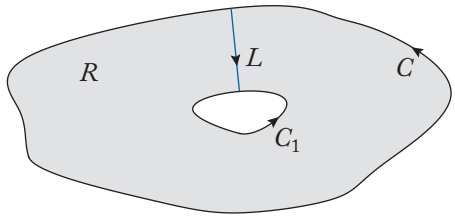


Figure 5.3: If f is analytic in R, the simple closed contour C may be deformed C_1. Path L is used in the proof of this result.

to the restriction given on the contour, it is appropriate to take the branch cut as the portion of the positive real axis $x \geq 2$. That is, we take the branch of $\log(w)$, with $w = 2z - 4$ to be

$$\log(w) = \ln|w| + i\arg(w), \qquad \text{where } 0 \leq \arg(w) < 2\pi.$$

Then we have

$$\begin{aligned} I &= \frac{3}{2}\left[\ln|4i-4| + i\arg(4i-4) - (\ln|-4i-4| + i\arg(-4i-4))\right] \\ &= \frac{3}{2}\left[\ln(\sqrt{32}) + i\frac{3\pi}{4} - \left(\ln(\sqrt{32}) + i\frac{5\pi}{4}\right)\right] = -\frac{i3\pi}{4}. \end{aligned}$$

Note that if the question had said that the contour did not cross the real axis with $x \leq 2$, then we would have taken a branch of $\log(w)$ such that $-\pi < \arg(w) \leq \pi$. In that case all of the above calculations would be the same except that $\arg(-4i-4)$ would be $-3\pi/4$ rather than $5\pi/4$. In that case the answer would end up being $i9\pi/4$.

5.1.3 PATH DEFORMATION

The third consequence of the Cauchy–Goursat theorem is, in some ways, analogous to the path independence theorem, except we now consider the case where the path is a closed contour.

Theorem 5.10 Path Deformation. *Let $f(z)$ be analytic in a region R bounded by two simple closed contours C and C_1, where C_1 lies inside C. Then*

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz.$$

Proof. Connect C to C_1 via a contour L as shown in Fig. 5.3. Then the contour $L - C_1 - L + C$ is the positively oriented boundary of a region in which f is analytic, hence, the Cauchy–Goursat

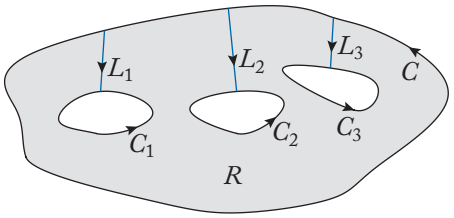


Figure 5.4: Path deformation of C to several separate contours C_i.

theorem holds for this contour giving

$$0 = \int_{L-C_1-L+C} f(z)\,dz = \int_L f(z)\,dz + \int_{-C_1} f(z)\,dz + \int_{-L} f(z)\,dz + \int_C f(z)\,dz$$
$$= \int_L f(z)\,dz - \int_{C_1} f(z)\,dz - \int_L f(z)\,dz + \int_C f(z)\,dz.$$

The integrals along L cancel and so the result follows. □

This theorem is important because it tells us that the value of $\int_C f(z)\,dz$ does not change when we deform C continuously to any other path C_1 provided the deformation always moves through points where f is analytic. The same concept is true for the path independence theorem; the path from p_1 to p_2 may be deformed to any other path provided the deformation moves through points where f is analytic, and, in this case, provided the end points of the path remain at p_1 and p_2.

Example 5.11 Compute $I = \int_C \frac{1}{z}\,dz$, where C is the square with vertices $\pm 2 \pm 2i$.

Instead of finding a parameterization of C, which has four separate arcs, since $1/z$ is analytic everywhere except at $z = 0$ we may deform the path C to the unit circle centered at 0 whose parameterization is $e^{i\theta}$, $0 \le \theta \le 2\pi$. Then

$$I = \int_0^{2\pi} \frac{1}{e^{i\theta}} i e^{i\theta}\,d\theta = \int_0^{2\pi} i\,d\theta = 2\pi i.$$

The deformation of path theorem also clearly extends to the case where there are n closed curves inside C as depicted in Fig. 5.4. In this case one can connect each closed inner curve C_k to C by a contour L_k, $1 \le k \le n$. Then

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz.$$

5.1.4 EXERCISES

5.1. Describe all closed contours C for which the Cauchy–Goursat theorem applies for the given function $f(z)$.

(a) $f(z) = \dfrac{1}{z}$ **(b)** $f(z) = z^{1/2}$ (principal branch)

(c) $f(z) = \dfrac{z}{e^z + 2i}$ **(d)** $f(z) = \dfrac{z+3}{\cosh(z/2)}$

(e) $f(z) = \dfrac{\sin z}{\mathrm{Log}(z) - i}$ **(f)** $f(z) = \mathrm{Log}(z^2 - 4z + 4)$

In (e) and (f) the principal branch of log is assumed, that is,

$$\mathrm{Log}(w) = \ln|w| + i\,\mathrm{Arg}(w), \quad -\pi < \mathrm{Arg}(w) \le \pi.$$

[Hint: For (f) let $w = z - 2$.]

5.2. Let C_1 and C_2 be the half circles, radius 1, centered at i going from 0 to $2i$ in the counter-clockwise and clockwise directions, respectively. Let $-C_1$ be the same path as C_1 but going from $2i$ to 0, and similarly let $-C_2$ be the same path as C_2 but traversed from $2i$ to 0.

(a) If $\displaystyle\int_{C_1} \sin(z)\,dz = 1 - \cosh(2)$, what is $I = \displaystyle\int_{C_2} \sin(z)\,dz$?

(b) What is $I = \displaystyle\oint_{C_1 - C_2} \frac{5}{(z-1)^2}\,dz$?

(c) If $\displaystyle\int_{C_1} \frac{2z}{z^2+1}\,dz = \ln(3) + i\pi$, what is $I = \displaystyle\int_{C_2} \frac{2z}{z^2+1}\,dz$? [Hint: It may be helpful to let $w = z^2 + 1$.]

5.3. Let C be the half circle, radius 1, centered at i going from $1 + i$ to $-1 + i$ in the counter-clockwise direction. Use the extension of the Fundamental theorem of calculus to compute

$$\int_C \frac{z}{z^2 + i}\,dz.$$

[Hint: It might help to let $w = z^2 + i$ and roughly sketch the contour C in the w-plane.]

5.4. For each of the following, specify some simply connected region R including the points p_1 and p_2 in which the given function is analytic, and then use antiderivatives to compute $\int_C f(z)\,dz$, where C is any contour joining p_1 to p_2 that is completely contained in R. If $f(z)$ is a multivalued function, choose any branch but be sure to clearly specify

which branch you are using and be careful to define it so that the branch cut does not enter your region R.

(a) $f(z) = z^{3/2}$, $p_1 = -1 - i\sqrt{3},\ p_2 = 1 + i\sqrt{3}$

(b) $f(z) = \dfrac{7z-2}{z^2+2z-8}$, $p_1 = -1 - i\sqrt{3},\ p_2 = -1 + 3i$

(c) $f(z) = \dfrac{\log(z)}{z}$, $p_1 = -4i,\ p_2 = -1/2$

5.5. Let $\text{Log}(z) = \ln|z| + i\,\text{Arg}(z)$ where $-\pi < \text{Arg}(z) \le \pi$.

(a) Find an antiderivative of $\text{Log}(z)$. [Hint: Use integration by parts on $1 \cdot \text{Log}(z)$.]

(b) Using the antiderivative from (a), compute $\int_i^{-i} \text{Log}(z)\,dz$.

(c) Let C be the path defined by $z(\theta) = e^{i\theta}$, $\pi/2 \le \theta \le 3\pi/2$. This path joins i to $-i$. Compute $\int_C \text{Log}(z)\,dz$ via the definition of the contour integral. If the answer you obtain here is different than the one obtained in (b), explain why.

5.6. For each of the following, use antiderivatives to compute $\int_C f(z)\,dz$. Express your answers in the form $x + iy$.

(a) $f(z) = \dfrac{1}{z}$, C is the portion of the circle radius three centered at the origin going from $-3i$ to $3i$ through the region $\text{Re}(z) \le 0$.

(b) $f(z) = (z-i)^{1/3}$, C is the straight line from $p_1 = -1 + 2i$ to $p_2 = -7i$.

5.7. Let Γ be the circle radius 2 centered at 1. Given that $\displaystyle\oint_\Gamma \frac{z-i+1}{z^2-2iz-1}\,dz = 2\pi$, specify all circles C centered at 0 such that $\displaystyle\oint_C \frac{z-i+1}{z^2-2iz-1}\,dz = 2\pi$.

5.2 THE CAUCHY INTEGRAL FORMULAS AND THEIR CONSEQUENCES

The Cauchy–Goursat theorem and the next theorem are the primary results of this chapter.

Theorem 5.12 Cauchy Integral Formulas[2]. *If f is analytic inside and on a simple closed contour C then for any point z_0 interior to C we have*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \qquad n = 0, 1, 2, \ldots, \tag{5.4}$$

where C is traversed in the positive sense.

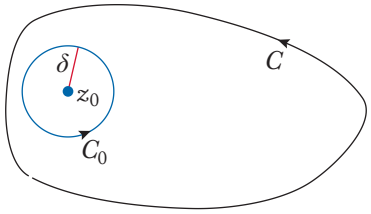


Figure 5.5: Circle C_0 centered at z_0 of radius δ inside C.

Proof. First we prove the theorem is true for the case $n = 0$. (Recall that by definition, $0! = 1$ and $f^{(0)} = f$.) Since z_0 is interior to C and since f is continuous, we may select a small circle C_0, of radius δ, centered at z_0 such that C_0 is itself interior to C (see Fig. 5.5) and such that the modulus of $f(z) - f(z_0)$ is small. To be precise, for any given $\epsilon < 0$ we may choose $\delta > 0$ such that $C_0 = \{z \mid |z - z_0| = \delta\}$ is interior to C and

$$|z - z_0| \leq \delta \qquad \Longrightarrow \qquad |f(z) - f(z_0)| < \epsilon.$$

Now $\dfrac{f(z)}{z - z_0}$ is analytic between C and C_0, thus by the deformation of path theorem

$$I = \oint_C \frac{f(z)}{z - z_0}\, dz = \oint_{C_0} \frac{f(z)}{z - z_0}\, dz.$$

We now add and subtract the same quantity from I to obtain

$$I = \oint_{C_0} \frac{f(z) - f(z_0)}{z - z_0}\, dz + \oint_{C_0} \frac{f(z_0)}{z - z_0}\, dz.$$

The latter of these can be evaluated directly using the parameterization $z(t) = z_0 + \delta e^{it}$ for C_0:

$$\oint_{C_0} \frac{f(z_0)}{z - z_0} = \int_0^{2\pi} \frac{f(z_0)}{\delta e^{it}} \delta i e^{it}\, dt = f(z_0) i \int_0^{2\pi} dt = f(z_0) 2\pi i.$$

For the former we have

$$\left| \oint_{C_0} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| \leq ML,$$

where L is the length of C_0 and M is the maximum of the modulus of the integrand on C_0. But clearly $L = 2\pi\delta$, and $M \leq \epsilon/\delta$, hence, $ML \leq 2\pi\epsilon$. Since ϵ is arbitrarily small, we conclude that $\displaystyle\oint_{C_0} \frac{f(z) - f(z_0)}{z - z_0}\, dz = 0$. Therefore, $I = f(z_0) 2\pi i$, which completes the proof for $n = 0$.

[2]Technically, the case $n = 0$ in the theorem is called the "Cauchy integral formula" and the case $n > 0$ is sometimes called the "generalized Cauchy integral formula" or the "Cauchy integral formula for derivatives."

We now prove the theorem for the cases $n > 0$ by using induction on n. Assume that (5.4) holds for $n = N - 1$, $N > 0$, then

$$\begin{aligned} f^{(N)}(z_0) &= \frac{d}{dz_0}\left[f^{(N-1)}(z_0)\right] = \frac{d}{dz_0}\left[\frac{(N-1)!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^N}\,dz\right] \\ &= \frac{(N-1)!}{2\pi i}\oint_C f(z)\frac{d}{dz_0}\left(\frac{1}{(z-z_0)^N}\right)dz \\ &= \frac{(N-1)!}{2\pi i}\oint_C f(z)\frac{N}{(z-z_0)^{N+1}}\,dz = \frac{N!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{N+1}}\,dz. \end{aligned}$$

Thus, by induction, the formula holds for all $n \geq 0$. □

This theorem is a really remarkable result for two reasons. First, it says that when f is analytic the values of f inside C are completely determined by the values of f *on* C. Second, it says that every analytic function has derivatives of *all* orders. Such a statement is certainly not true for real-valued functions of a real variable. For example, $f(x) = x^{4/3}$ has derivative $f'(x) = \frac{4}{3}x^{1/3}$ which is defined and continuous for all x, but $f''(x) = \frac{4}{9}x^{-2/3}$ is not defined at $x = 0$.

The fact that we have now shown that analytic functions have derivatives of all orders, brings us back to the discussion of the Cauchy–Goursat theorem. Recall that Cauchy proved the theorem (as did we) by imposing the extra assumption that f' was continuous. Goursat however, gave a proof without this additional restriction. The result of the Cauchy–Goursat theorem was used to prove the path deformation theorem, which in turn was used in the proof of the Cauchy integral formulas, which tell us that analytic functions not only have continuous derivatives, but in fact differentiable derivatives. If Goursat had not given his proof then it would still be possible that there exists some analytic function (which means it has a derivative) but whose derivative is not continuous. In that case, Cauchy's proof of the Cauchy–Goursat theorem would not apply to that function and so you would not be able to deform the path and would not be able to prove the Cauchy integral formulas for such a function. But as it is, Goursat's proof of the Cauchy–Goursat theorem means that the Cauchy integral formulas most hold for any analytic function, hence, all analytic functions have derivatives of all orders and so all derivatives are continuous. Thus, Cauchy's extra assumption was valid.

The Cauchy integral formulas are often used in reverse, that is, rather than computing an integral in order to find the value of a derivative, one usually applies the formula in the other direction, obtaining the value of an integral by computing the value of a derivative, as the next three examples illustrate.

Example 5.13 Compute $I = \oint_C \frac{e^{iz}}{z^3}\,dz$, where C is the circle $|z| = 2$.

If we define $f(z) = e^{iz}$ and $z_0 = 0$, then by the Cauchy integral formulas

$$I = \oint_C \frac{f(z)}{(z-0)^3}\,dz = \frac{2\pi i}{2!} f''(0) = \frac{2\pi i}{2}\left(i^2 e^{i0}\right) = -\pi i.$$

Example 5.14 Compute $I = \oint_C \dfrac{\cos(z\pi/3)}{z^2-1}\,dz$, where C is the positively oriented boundary of the rectangle with vertices $0 \pm i$ and $2 \pm i$.

We note that $z^2 - 1 = (z-1)(z+1)$ and that 1 is interior to C but -1 is not. We therefore write

$$I = \oint_C \frac{\cos(z\pi/3)/(z+1)}{z-1}\,dz.$$

Now $f(z) = \cos(z\pi/3)/(z+1)$ is analytic inside and on C, and $z_0 = 1$ is interior to C, hence, by the Cauchy integral formula

$$I = 2\pi i f(1) = 2\pi i \left(\frac{\cos(\pi/3)}{2}\right) = \pi i \cos(\pi/3) = \frac{i\pi}{2}.$$

Example 5.15 Compute $I = \oint_C \dfrac{z^3 - 2i}{z^2+1}\,dz$, where C is the circle $|z| = 2$.

Here we have $z^2 + 1 = (z-i)(z+i)$ but both $+i$ and $-i$ are inside C. Since these are the only two singularities of the integrand we deform the path C to two small circles C_1 and C_2 around i and $-i$, respectively. Then

$$\begin{aligned} I &= \oint_{C_1} \frac{z^3-2i}{(z-i)(z+i)}\,dz + \oint_{C_2} \frac{z^3-2i}{(z-i)(z+i)}\,dz \\ &= \oint_{C_1} \frac{(z^3-2i)/(z+i)}{z-i}\,dz + \oint_{C_2} \frac{(z^3-2i)/(z-i)}{z+i}\,dz. \end{aligned}$$

The numerator of each integrand on the right is analytic inside the respective contour, so we may apply the Cauchy integral formula to each of the above integrals giving

$$I = 2\pi i \left(\frac{i^3-2i}{i+i}\right) + 2\pi i \left(\frac{(-i)^3-2i}{-i-i}\right) = \pi(-3i) - \pi(-i) = -2\pi i.$$

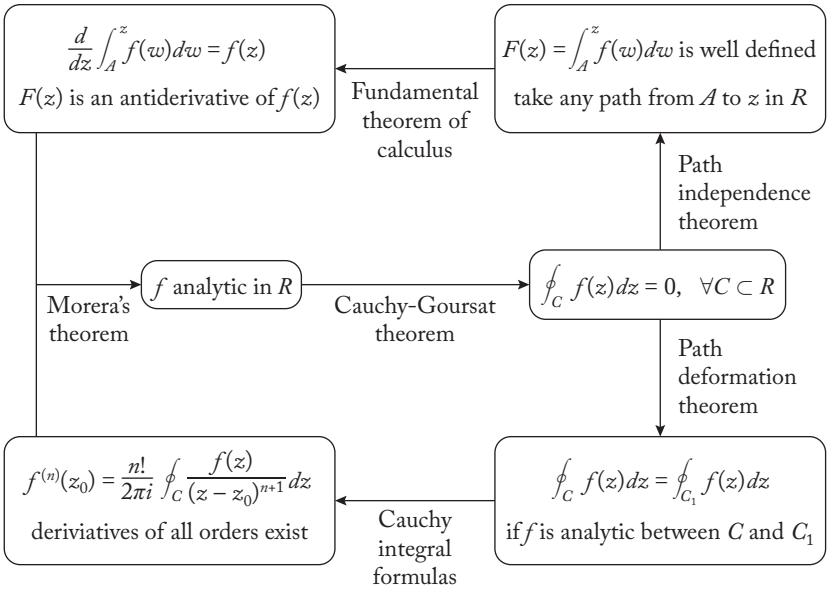

$\frac{d}{dz}\int_A^z f(w)dw = f(z)$
$F(z)$ is an antiderivative of $f(z)$
Fundamental theorem of calculus
$F(z) = \int_A^z f(w)dw$ is well defined
take any path from A to z in R
Path independence theorem
Morera's theorem
f analytic in R
Cauchy-Goursat theorem
$\oint_C f(z)dz = 0, \quad \forall C \subset R$
Path deformation theorem
$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$
deriviatives of all orders exist
Cauchy integral formulas
$\oint_C f(z)dz = \oint_{C_1} f(z)dz$
if f is analytic between C and C_1

Figure 5.6: Relationship between contour integral theorems.

5.2.1 MORERA'S THEOREM

The Cauchy integral formulas have many consequences, and the next sections will be devoted to enumerating them. However, in this section we will state and prove Morera's theorem, which is the converse of the Cauchy–Goursat theorem.[3]

Theorem 5.16 Morera's Theorem. *If f is continuous in a simply connected region R and if $\oint_C f(z)\,dz = 0$ for every simple closed contour C in R then f is analytic in R.*

Proof. We have already shown that $\oint_C f(z)\,dz = 0$ for all simple closed contours C in R implies path independence in R and this implies the existence of an antiderivative $F(z)$ such that $F'(z) = f(z)$ in R. Thus, by definition $F(z)$ is analytic in R. Then, by the Cauchy integral formulas we have that F must have derivatives of all orders and in particular $F''(z) = f'(z)$ exists. Hence, f is analytic. □

Figure 5.6 displays the relationship between the theorems we have covered so far in this chapter. The boxes represent statements and the arrows correspond to theorems. The figure is intended to convey these relationships based on the presentation of these theorems and their

[3]Giacinto Morera (1856–1909) was an Italian engineer and mathematician.

proofs that we have used in this text. The arrows point to the result of the corresponding theorems, however they emerge not from the hypothesis of the theorem but from the key element needed in the theorem's proof. Indeed, for each of the theorems mentioned in the figure except Morera's theorem, the hypothesis is that f is analytic in a region R, which is the middle box on the left. Consider, for example, the Path independence theorem. The key feature needed in its proof is the statement that the integral of f around all closed contours in R is zero, which is the middle box on the right. But the middle box on the right follows from the middle box on the left by the Cauchy–Goursat theorem.

5.2.2 CAUCHY'S INEQUALITY

Theorem 5.17 Cauchy's Inequality. *If f is analytic inside and on a circle C of radius R centered at $z = z_0$ then*

$$\left| f^{(n)}(z_0) \right| \leq \frac{Mn!}{R^n}, \qquad n = 0, 1, 2, \ldots,$$

where $|f(z)| \leq M$ on C.

Proof. By Cauchy's integral formulas we have

$$\left| f^{(n)}(z_0) \right| = \left| \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \right|.$$

But on C we have $|f(z)| \leq M$ and $\left|(z - z_0)^{n+1}\right| = R^{n+1}$, and the arc length of C is $2\pi R$, thus

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{2\pi i} \left(\frac{M}{R^{n+1}} \right) (2\pi R) = \frac{Mn!}{R^n}.$$

□

5.2.3 LIOUVILLE'S THEOREM

Joseph Liouville (1809–1882) was a French mathematician whose name is attached to the following theorem.

Theorem 5.18 Liouville's Theorem. *If f is an entire function (analytic on all of $\mathbb{C}$) and if it is bounded, that is, there exists a positive number M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then f is a constant.*

Proof. Choose a point z_0 and a circle radius R centered at z_0, and put $n = 1$ in Cauchy's inequality, then

$$|f'(z_0)| \leq \frac{M}{R}.$$

Now take the limit as $R \to \infty$. (This is possible since f is analytic everywhere.) We conclude that $|f'(z_0)| = 0$, hence, $f'(z_0) = 0$. This is true for any z_0 in $\mathbb{C}$, which implies that $f(z)$ is a constant. □

5.2.4 FUNDAMENTAL THEOREM OF ALGEBRA

With Liouville's theorem in hand we can now give a proof of the Fundamental theorem of algebra.

Theorem 5.19 Fundamental Theorem of Algebra. *Every polynomial* $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ *with* $n \geq 1$ *and* $a_n \neq 0$ *has at least one zero, that is, there exists and least one value* z_0 *such that* $P(z_0) = 0$.

Proof. Suppose $P(z)$ has no zero, then $f(z) = \dfrac{1}{P(z)}$ is an entire function. If we can show that $f(z)$ is bounded, then by Liouville's theorem, $f(z)$ must be a constant, which is clearly a contraction since $P(z)$ is not constant. Hence, the assumption that $P(z)$ has no zero must be false. The remainder of the proof is simply to show that $f(z)$ is bounded.

Consider a circle radius $R > 1$ centered at the origin, and write $P(z) = z^n(a_n + w)$, where

$$w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}.$$

The function $|f|$ is continuous on the closed bounded set $|z| \leq R$, hence, by Theorem 2.2, $|f|$ must be bounded on that set. For $|z| > R$ we have

$$|P(z)| > R^n |a_n + w| \geq R^n (|a_n| - |-w|),$$

which we obtained via the triangle inequality: $|A - B| \geq |A| - |B|$. But, since $R > 1$,

$$|-w| = |w| = \left| \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}} \right| \leq \sum_{k=0}^{n-1} \left| \frac{a_k}{z^{n-k}} \right| \leq \sum_{k=0}^{n-1} \frac{|a_k|}{R^{n-k}} \leq n \frac{A}{R}, \qquad \text{where } A = \max_{0 \leq k \leq n-1} |a_k|.$$

We now choose $R > \max\left(1, \frac{2nA}{|a_n|}\right)$, then $|-w| \leq \frac{|a_n|}{2}$ so that

$$|P(z)| > R^n \left(|a_n| - \frac{|a_n|}{2} \right) = \frac{R^n |a_n|}{2}, \qquad \text{for } |z| > R.$$

Therefore,

$$|f(z)| < \frac{2}{|a_n|\, R^n}, \qquad \text{for } |z| > R.$$

We conclude that $|f(z)|$ is bounded on all of $\mathbb{C}$. This completes the proof. □

The Fundamental theorem of algebra (FTA) is often stated as the following corollary.

Corollary 5.20 Alternate FTA. *Every polynomial equation* $P(z) = a_0 + a_1 z + \cdots + a_n z^n = 0$, *with* $a_n \neq 0$, *has exactly n roots, some possibly repeated.*

Proof. Apply the FTA to $P(z)$ to find a zero z_1. Then we may factor out this zero and write

$$P(z) = (z - z_1)\left[b_0 + b_1 z + \cdots + b_{n-1} z^{n-1}\right],$$

where $b_{n-1} = a_n \neq 0$, and the remaining b_k are appropriate combinations of the a_k. We may now apply the FTA to the degree $n-1$ polynomial in brackets above to find another zero z_2. Continuing in this manner we have

$$P(z) = (z - z_1)(z - z_2)\cdots(z - z_n)a_n.$$

Thus, each of zeros $z_1, z_2, \ldots, z_n$ are roots of the equation $P(z) = 0$. □

5.2.5 GAUSS' MEAN VALUE THEOREM

Theorem 5.21 Gauss' Mean Value Theorem. *If* f *is analytic inside and on a circle* C *centered at* z_0 *with radius* R *then* $f(z_0)$ *is the mean of the values of* $f(z)$ *on* C, *that is,*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\, dt.$$

Proof. A parameterization for C is $z(t) = z_0 + Re^{it}$, $0 \leq t \leq 2\pi$. From Cauchy's integral formula we have

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + Re^{it})}{Re^{it}} iRe^{it}\, dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\, dt.$$

□

5.2.6 MAXIMUM MODULUS THEOREM

This next theorem will have significant consequences for harmonic functions. Recall that a domain is an open connected set.

Theorem 5.22 Maximum Modulus Theorem. *If f is analytic in a domain D then either $f(z)$ is constant in D or $|f(z)|$ does not attain a maximum in D.*

The domain here may be multiply connected. Before proving this theorem we state a corollary in the case where the domain is bounded.

Corollary 5.23 *If f is analytic in a bounded domain D with boundary C, and if f is continuous in $\overline{D} = D \cup C$, then either $f(z)$ is constant in $\overline{D}$ or $|f(z)|$ attains its maximum on the boundary C.*

Proof of Theorem 5.22: Suppose the maximum of $|f(z)|$ occurs at $z_0 \in D$. Then by Gauss' mean value theorem, for some R, sufficiently small so that the circle radius R centered at z_0 is also in D, we have

$$|f(z_0)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0 + Re^{it})\,dt\right| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0 + Re^{it})\right|\,dt.$$

Suppose $\left|f(z_0 + Re^{it_0})\right| < |f(z_0)|$ for some fixed t_0. Then, by continuity of f, $\left|f(z_0 + Re^{it})\right| < |f(z_0)|$ on some interval $[t_0 - \delta, t_0 + \delta]$, with $\delta > 0$. Since for no other t can we have $\left|f(z_0 + Re^{it})\right| > |f(z_0)|$, we conclude that

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0 + Re^{it})\right|\,dt < |f(z_0)|,$$

which is a contradiction. Therefore, either $f(z)$ is constant or $|f(z)|$ does not attain its maximum in D. □

Proof of Corollary 5.23: Inside D the theorem holds so either $f(z)$ is constant in D or $|f(z)|$ does not attain its maximum in D. If f is constant in D then by continuity it is constant on $\overline{D}$. In the case that f is not constant, since $\overline{D}$ is closed and bounded, by Theorem 2.2 the continuous function $|f(z)|$ must attain its maximum in $\overline{D}$. Since the maximum cannot be in D it must be on C. □

Since the real and imaginary parts of an analytic function are harmonic functions, Theorem 5.22 places constraints on harmonic functions. Before stating and proving that result we first prove that harmonic conjugates always exist.

Theorem 5.24 Existence of Harmonic Conjugates. *If u is harmonic in a simply connected domain D, there exists a harmonic function v in D such that $f = u + iv$ is analytic in D.*

Proof. Choose a point $(x_0, y_0) \in D$ and select a value v_0. Define $v(x_0, y_0) = v_0$ and for other points in D define

$$v(x, y) = \int_C -u_t(s,t)\,ds + u_s(s,t)\,dt,$$

where C is some path from (x_0, y_0) to (x, y) that is completely in D. For the above definition to be sensible, it must be independent of the particular path C in D that is chosen. For any two such paths C_1 and C_2, define $C_3 = C_1 - C_2$, and let R be the region bounded by C_3. Then by Green's theorem (Theorem 5.1 with $P = -u_t$ and $Q = u_s$) we have

$$\oint_{C_3} -u_t(s,t)\,ds + u_s(s,t)\,dt = \iint_R ((u_s)_s - (-u_t)_t)\,dsdt = \iint_R u_{ss} + u_{tt}\,dsdt = 0,$$

since u is harmonic in D. Therefore,

$$\int_{C_1} -u_t(s,t)\,ds + u_s(s,t)\,dt = \int_{C_2} -u_t(s,t)\,ds + u_s(s,t)\,dt,$$

showing that v is well defined. Due to path independence, a path from (x_0, y_0) to $(x + \Delta x, y)$ can be taken as C plus the straight line L from (x, y) to $(x + \Delta x, y)$ on which $t = y$ and $dt = 0$, thus from the definition of the derivative,

$$\begin{aligned} v_x(x,y) &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_{C+L} -u_t\,ds + u_s\,dt - \int_C u_t\,ds + u_s\,dt\right] \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\int_L -u_t\,ds + u_s\,dt = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\int_x^{x+\Delta x} -u_y(s,y)\,ds = -u_y(x,y). \end{aligned}$$

Similarly, we can show $v_y = u_x$. Hence, the CR equations hold, and since the partial derivatives of u are continuous so must be the partial derivatives of v. Therefore, $f = u + iv$ is analytic in D. □

The arbitrary value v_0 in the above proof shows that harmonic conjugates are only unique up to a constant. It is also worth remarking that harmonic conjugates also exist if the domain D is multiply-connected. In this case the value of v may not be independent of path since two paths going opposite ways around a hole in D may yield different values for v. In other words, v may be a multivalued function. A simple example is $u = \frac{1}{2}\ln(x^2 + y^2)$, which has harmonic conjugate

$$v = \arctan\left(\frac{y}{x}\right) + 2\pi k + v_0, \qquad k \in \mathbb{Z},$$

for any constant v_0. In this case the function $f = u + iv$ is an appropriate branch of $\log(z) + v_0$.

We are now prepared to see how the Maximum modulus theorem constrains harmonic functions.

Theorem 5.25 Maximum Principle for Harmonic Functions. *If u is a harmonic function in a domain D and if it attains its maximum or minimum value at some point $z_0 \in D$, then u is constant in D.*

Proof. First suppose D is simply connected. Given u, construct the harmonic conjugate v (which, by Theorem 5.24 is always possible) so that $f = u + iv$ is analytic. Now consider $g = e^f$. This function is analytic also and

$$|g| = \left|e^f\right| = \left|e^{u+iv}\right| = |e^u|\,|e^{iv}| = |e^u|.$$

Clearly, $|g|$ is a maximum when u is at a maximum. Therefore, by Theorem 5.22, if u attains a maximum on D then g and therefore u must be constant. The result for the minimum follows similarly by considering $g = e^{-f}$, which is also analytic.

Now in the case where D is multiply connected, one can add in contours L_i from the outer boundary to each inner boundary as in Fig. 5.4 to produce a simply connected region on which the theorem must hold. However, because the precise location of the L_i is arbitrary, the maximum and minimum values of u cannot occur on any of the L_i, since if they did, then a slight movement of the L_i would produce a simply connected region where the maximum or minimum values occurred in the interior, violating the above proof. □

The maximum principle for harmonic functions can be viewed simply as a result about functions on $\mathbb{R}^2$. Often, as with the maximum modulus theorem, it is stated slightly more strongly in the case when the domain is bounded.

Corollary 5.26 *If u is a harmonic function on a bounded domain D with boundary C and continuous on $\overline{D}$, then u attains its maximum and minimum values on C.*

5.2.7 MINIMUM MODULUS THEOREM

There is also a similar theorem for the minimum modulus of an analytic function, but it requires the additional restriction that $f(z)$ cannot be zero.

Theorem 5.27 Minimum Modulus Theorem. *If f is analytic in a domain D and $f(z) \neq 0$ in D, then either $f(z)$ is constant in D or $|f(z)|$ does not attain a minimum in D.*

Again we state a corollary before proving both.

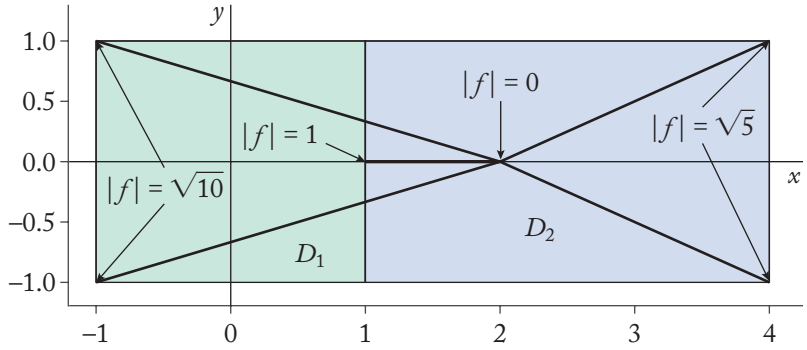


Figure 5.7: Domains and moduli for Example 5.29; $|f|$ is the distance to $z = 2$.

Corollary 5.28 *If f is analytic in a bounded domain D with boundary C and if f is continuous in $\overline{D}$ and $f(z) \neq 0$ in D, then either $f(z)$ is constant in $\overline{D}$ or $|f(z)|$ attains its minimum on the boundary C.*

Proof of Theorem 5.27: Let $g(z) = \dfrac{1}{f(z)}$. Then, since $f(z) \neq 0$ in D, $g(z)$ is analytic in D and the Maximum modulus theorem applies. Hence, either $g(z)$ is constant in D or $|g(z)|$ does not attain a maximum in D. Therefore, either $f(z)$ is constant in D or $|f(z)|$ does not attain its minimum in D. □

Proof of Corollary 5.28: This proof is the same as the proof of Corollary 5.23 with the word "maximum" replaced with "minimum." □

Example 5.29 Let D_1 be the interior of the square with vertices $\pm 1 \pm i$, and let D_2 be the interior of the rectangle with vertices $1 \pm i$ and $\pm 4 \pm i$. Find the maximum and minimum modulus of the function $f(z) = z - 2$ on $\overline{D_1}$ and on $\overline{D_2}$.

The situation is illustrated in Fig. 5.7. For this function, $|f|$ is simply the distance to the point $z = 2$. Both the maximum and the minimum modulus theorems apply for D_1, and it is not difficult to see that the maximum of $|f|$ on $\overline{D_1}$ is $\sqrt{10}$, which occurs at the two corners $z = -1 \pm i$, while the minimum of $|f|$ is 1, which occurs at $z = 1$ along the right boundary of D_1. For D_2, the maximum modulus theorem applies but not the minimum modulus theorem since $f(z) = 0$ at $z = 2$, which is inside D_2. This is clearly the location of the minimum of $|f|$. The maximum of $|f|$ is $\sqrt{5}$, which occurs at the two corners $z = 4 \pm i$ of D_2.

5.2.8 POISSON'S INTEGRAL FORMULAS FOR THE CIRCLE AND HALF-PLANE

The next two consequences of the Cauchy integral formula are attributed to Siméon Poisson (1781–1840).

Theorem 5.30 Poisson's Integral Formula for the Circle. *Let f be analytic inside and on a circle C defined by $|z| = R$. Then if $r < R$ and $z = re^{i\theta}$ we have*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})}{\left|Re^{i\phi} - re^{i\theta}\right|^2}\, d\phi. \tag{5.5}$$

Proof. By Cauchy's integral formula

$$f(re^{i\theta}) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - re^{i\theta}}\, dz. \tag{5.6}$$

The point $\frac{R^2}{r}e^{i\theta}$ lies outside C, hence,

$$0 = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - \frac{R^2}{r^2}e^{i\theta}}\, dz,$$

since the integrand is analytic inside C. Subtracting this quantity from both sides of (5.6) we obtain

$$\begin{aligned} f(re^{i\theta}) &= \frac{1}{2\pi i}\oint_C \frac{f(z)\left(z - \frac{R^2}{r}e^{i\theta}\right) - f(z)\left(z - re^{i\theta}\right)}{\left(z - re^{i\theta}\right)\left(z - \frac{R^2}{r}e^{i\theta}\right)}\, dz \\ &= \frac{1}{2\pi i}\oint_C \frac{\left(r - \frac{R^2}{r}\right) f(z)e^{i\theta}}{z^2 - ze^{i\theta}\left(r + \frac{R^2}{r}\right) + R^2e^{i2\theta}}\, dz \\ &= \frac{1}{2\pi i}\oint_C \frac{\left(R^2 - r^2\right) f(z)e^{i\theta}\left(\frac{-1}{r}\right)}{z^2 - ze^{i\theta}\frac{1}{r}\left(r^2 + R^2\right) + R^2e^{i2\theta}}\, dz. \end{aligned}$$

But on C we have $z = Re^{i\phi}$, so

$$\begin{aligned} f(re^{i\theta}) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})e^{i\theta}\left(\frac{-1}{r}\right)}{R^2e^{i2\phi} - \frac{R}{r}e^{i(\theta+\phi)}(r^2 + R^2) + R^2e^{i2\theta}} Rie^{i\phi}\, d\phi \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})(-i)}{Rre^{i(\phi-\theta)} - (r^2 + R^2) + Rre^{i(\theta-\phi)}}\, d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi. \end{aligned}$$

To see that the second equality in (5.5) is true, note that the denominator is

$$\begin{aligned}\left|Re^{i\phi}-re^{i\theta}\right|^2 &= \left(Re^{i\phi}-re^{i\theta}\right)\left(Re^{-i\phi}-re^{-i\theta}\right) = R^2 - Rre^{i(\phi-\theta)} - Rre^{i(\theta-\phi)} + r^2 \\ &= R^2 - 2Rr\cos(\theta-\phi) + r^2.\end{aligned}$$

□

If we write $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ then taking the real part of (5.5) it immediately follows that

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)u(R,\phi)}{R^2-2Rr\cos(\theta-\phi)+r^2}\,d\phi, \tag{5.7}$$

and a similar equation holds for the imaginary part v. Since u is a harmonic function, the above formula may be used to solve Laplace's equation on a disk with the value of u being prescribed on the boundary.

Example 5.31 Consider a disk of radius 1 that is held at temperature 3 along one half of the boundary and temperature 0 on the other half, that is,

$$u(1,\theta) = \begin{cases} 3 & \text{if } 0 \le \theta \le \pi, \\ 0 & \text{if } \pi < \theta < 2\pi. \end{cases}$$

Find the steady-state temperature $u(r,\theta)$ in the disk.

The temperature u satisfies the heat equation $u_t = \triangle u$. At steady state, the time derivative is zero, so u satisfies Laplace's equation. This means u is a harmonic function and we may apply Eq. (5.7). Choose a circle radius $R < 1$ and take the limit as R approaches 1.

$$\begin{aligned} u(r,\theta) &= \lim_{R\to 1}\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)u(R,\phi)}{R^2-2Rr\cos(\theta-\phi)+r^2}\,d\phi \\ &= \frac{1}{2\pi}\left[\int_0^{\pi}\frac{(1-r^2)3}{1+r^2-2r\cos(\theta-\phi)}\,d\phi + \int_{\pi}^{2\pi}\frac{(1-r^2)(0)}{1+r^2-2r\cos(\theta-\phi)}\,d\phi\right] \\ &= \frac{3}{2\pi}\int_0^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}\,d\phi. \end{aligned}$$

This integral is a difficult exercise in real variable calculus involving a substitution $t = (\theta-\phi)/2$, the use of a double angle formula and other trigonometric identities, a splitting of the range of integration, and significant algebraic manipulation. The result is

$$u(r,\theta) = \begin{cases} \frac{3}{\pi}\left[\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\pi-\theta}{2}\right)\right) + \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta}{2}\right)\right)\right] & \text{if } 0 \le \theta \le \pi, \\ \frac{3}{\pi}\left[\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\pi-\theta}{2}\right)\right) + \arctan\left(\frac{1+r}{1-r}\tan\left(\frac{\theta}{2}\right)\right) + \pi\right] & \text{if } \pi < \theta < 2\pi. \end{cases}$$

As necessary, in the limit as $r \to 1^-$ the above function approaches 3 if $0 \le \theta \le \pi$ and approaches 0 if $\pi < \theta < 2\pi$. (See Example 8.9 for another way to solve this problem.)

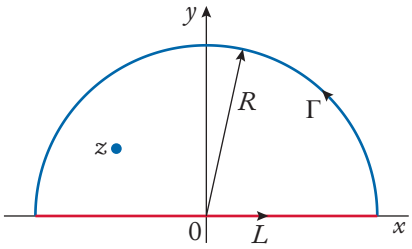


Figure 5.8: Contour $C = L + \Gamma$ used in the proof of Poisson's integral formula for the half-plane.

The next theorem is an analogous result for the half-plane. In this case, to prove the theorem, a half-circle contour is chosen, along which, in a certain limit, the integral is zero.

Theorem 5.32 Poisson's Integral Formula for the Half-Plane. *Let f be analytic in the upper half-plane and let $z = x + iy$ be any point in this upper half-plane, that is, $y > 0$. Also assume that $\lim_{|z|\to\infty} |f(z)| \le M\,|z|^a$ for some real number $a < 1$. Then*

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y f(\xi)}{(\xi - x)^2 + y^2}\, d\xi = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y f(\xi)}{|\xi - z|^2}\, d\xi. \tag{5.8}$$

Proof. Choose $R > |z|$. Consider the contour C made up of the line segment L from $-R$ to $+R$ on the real axis and the semi-circle Γ of radius R centered at 0 and in the upper half-plane, as shown in Fig. 5.8. Since f is analytic inside and on C, by the Cauchy integral formula we have

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w - z}\, dw. \tag{5.9}$$

The point $\overline{z}$ is outside the contour C, hence,

$$0 = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w - \overline{z}}\, dw,$$

since the integrand is analytic inside C. Subtracting this quantity from both sides of (5.9) we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(w)(w-\overline{z}) - f(w)(w-z)}{(w-z)(w-\overline{z})}\,dw \\ &= \frac{1}{2\pi i}\oint_C \frac{i2yf(w)}{w^2 - 2xw + z\overline{z}}\,dw \\ &= \frac{1}{2\pi i}\left[\int_L \frac{i2yf(w)}{w^2 - 2xw + z\overline{z}}\,dw + \int_\Gamma \frac{i2yf(w)}{w^2 - 2xw + z\overline{z}}\,dw\right]. \end{aligned}$$

For the integral along Γ, we have $w = Re^{i\phi}$, $0 \le \phi \le \pi$, and the arc length of Γ is πR. Therefore,

$$\begin{aligned} \lim_{R\to\infty}\left|\int_\Gamma \frac{i2yf(w)}{w^2 - 2xw + z\overline{z}}\,dw\right| &\le \lim_{R\to\infty}\max_{0\le\phi\le\pi}\left|\frac{i2yf(Re^{i\phi})}{R^2e^{i2\phi} - 2xRe^{i\phi} + z\overline{z}}\right|\pi R \\ &\le \lim_{R\to\infty}\max_{0\le\phi\le\pi}\frac{2|y|MR^a}{R^2\left|1 - \frac{2x}{Re^{i\phi}} + \frac{|z|^2}{R^2e^{i2\phi}}\right|}\pi R \\ &= \lim_{R\to\infty}\frac{2|y|M\pi}{R^{1-a}} = 0, \end{aligned}$$

since $a < 1$. Therefore, in the limit as $R \to \infty$ only the integral along L remains. On L we have $w = \xi$, thus

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i2yf(\xi)}{\xi^2 - 2x\xi + x^2 + y^2}\,d\xi = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yf(\xi)}{(\xi - x)^2 + y^2}\,d\xi.$$

The second equality in the theorem statement follows from the fact that

$$|\xi - z|^2 = (\xi - z)(\xi - \overline{z}) = \xi^2 - \xi(z + \overline{z}) + z\overline{z} = \xi^2 - 2\xi x + x^2 + y^2 = (\xi - x)^2 + y^2.$$

□

As in the case for the circle, we may take the real and imaginary parts of (5.8) to get

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yu(\xi, 0)}{(\xi - x)^2 + y^2}\,d\xi,$$

and a similar formula for $v(x, y)$, where $f = u + iv$.

Example 5.33 Find a solution of Laplace's equation on the half-plane $y > 0$ that satisfies

$$u(x, 0) = \begin{cases} 1 & \text{if } |x| < 5, \\ 0 & \text{otherwise.} \end{cases}$$

Using Poisson's real part of the integral formula for the half-plane, we have

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yu(\xi,0)}{(\xi-x)^2+y^2}\,d\xi = \frac{1}{\pi}\int_{-5}^{5} \frac{y}{(\xi-x)^2+y^2}\,d\xi.$$

Using the substitution $t = (\xi - x)/y$ this becomes

$$\begin{aligned} u(x,y) &= \frac{1}{\pi}\int_{\frac{-5-x}{y}}^{\frac{5-x}{y}} \frac{1}{t^2+1}\,dt = \frac{1}{\pi}\Big[\arctan(t)\Big]_{\frac{-5-x}{y}}^{\frac{5-x}{y}} \\ &= \frac{1}{\pi}\left[\arctan\left(\frac{5-x}{y}\right) - \arctan\left(\frac{-5-x}{y}\right)\right]. \end{aligned}$$

5.2.9 EXERCISES

5.8. Let C be the circle of radius 4 centered at the origin. Use the Cauchy integral formula to compute

$$\textbf{(a)}\ \oint_C \frac{\sin(5z-i)}{(z-i)^3}\,dz, \qquad \textbf{(b)}\ \oint_C \frac{\mathrm{Log}(z+7-5i)}{z^2+(6-2i)z+8-6i}\,dz,$$

where $\mathrm{Log}(z)$ is the principal branch of the logarithm.

5.9. Let C be the circle $|z| = 4$. Show that $\dfrac{1}{2\pi i}\displaystyle\oint_C \frac{e^{zt}}{z^2+1}\,dz = \sin t$.

5.10. Let $I_n = \displaystyle\oint_C \frac{\mathrm{Log}(z-4+i)}{(z-3)^{n+1}}\,dz$, where C is a simple closed contour with 3 on its inside, n is a nonnegative integer, and $\mathrm{Log}(w)$ is the principal branch, namely, $\mathrm{Log}(w) = \log|w| + i\arg(w)$, with $-\pi < \arg(w) \le \pi$.

(a) If one wishes to use the Cauchy integral formula to compute I_n what restrictions, if any, need to be applied to C?

(b) Using the Cauchy integral formula, compute I_6.

5.11. Let C be the unit circle centered at the origin. First show using Cauchy's integral formula, that

$$\oint_C \frac{e^{az}}{z}\,dz = 2\pi i,$$

for any real constant a. Now parameterize C in terms of θ and write the above integral in terms of an integral in θ to derive the integration formula

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi.$$

5.12. Let f be an entire function such that $|f(z)| \leq A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = az$ where a is a complex constant. [Hint: Use Cauchy's inequality for f'' with a circle radius r centered at z_0, to show that $f'' = 0$. First you will need to show that the constant M in Cauchy's inequality is less than or equal to $A(|z_0| + r)$. A different way to solve this problem is to show that the limit as $z \to 0$ of $f(z)/z$ exists, and then define a function g as $g(z) = f(z)/z$ when $z \neq 0$ and $g(0)$ equals the limit. Then argue that g is entire and apply Liouville's theorem to g.]

5.13. Suppose that $f(z) = u(x, y) + iv(x, y)$ is entire and that u is bounded from above, that is, $u(x, y) \leq M$ for all (x, y) in the plane, where M is a finite constant. Show that $u(x, y)$ must be a constant throughout the plane. [Hint: Use Liouville's theorem and consider the function $g = e^f$.] Is the result still true if u is bounded from below $(u(x, y) \geq M)$?

5.14. Evaluate each of the following integrals using Gauss' mean value theorem.

$$\textbf{(a)} \int_0^{2\pi} \cos^4\left(\pi/3 + 3e^{i\theta}\right) d\theta, \qquad \textbf{(b)} \int_0^{2\pi} \frac{1 + e^{i\theta}}{3 + e^{i\theta}}\, d\theta.$$

5.15. Let $f(z) = z^2 + 4z$. Find the maximum and minimum values of $|f(z)|$ on the disk $|z| \leq 3$. Also indicate where on the disk these maximum and minimum values occur.

5.16. Let $f(z) = \log(z)$, where the branch cut is on the positive x-axis and $0 \leq \arg(z) < 2\pi$. Find the maximum and minimum of $|f(z)|$ on the semi-annular region $R = \left\{z \in \mathbb{C} \,:\, 1 \leq |z| \leq 2,\ \frac{\pi}{2} \leq \arg(z) \leq \frac{3\pi}{2}\right\}$.

5.17. The function $u(x, y) = 2x(1 - y)$ is harmonic. Find the maximum and minimum of u on the disk radius two centered at the origin and indicate where these extrema occur.

5.18. The steady-state temperature, $u(x, y)$, for a two-dimensional material satisfies Laplace's equation, i.e., it is a harmonic function. Consider a material covering the entire upper half-plane, $y \geq 0$, where the boundary temperature is given by

$$u(x, 0) = \begin{cases} 40 - |x| & \text{if } |x| \leq 20, \\ 0 & \text{otherwise.} \end{cases}$$

Use Poisson's formula to find $u(x, y)$ on the upper half-plane.

5.3 COUNTING ZEROS AND POLES

In this section we consider several theorems that are concerned with the number of zeros and poles of a function within a region. These theorems rely on the fact that near a zero/pole z_0 of order m the argument of a function changes like $\pm m \arg(z - z_0)$.

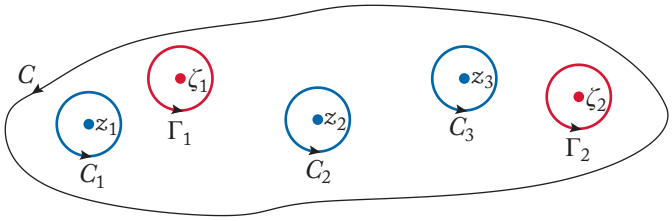


Figure 5.9: Deformation of the path C to circles C_j around zeros and circles Γ_k around poles.

5.3.1 ARGUMENT THEOREM

Theorem 5.34 Argument Theorem. *Let f be analytic inside and on a simple closed contour C except for a finite number of poles inside C. Suppose f has a finite number of zeros inside C and that $f(z) \neq 0$ on C. Then*

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N - P,$$

where N is the number of zeros and P the number of poles of f inside C, respectively, counting order (for example, a pole order two counts 2, etc.).

Proof. By the assumptions of the theorem, the integrand is analytic except at poles of f and at zeros of f. There are a finite number of such. (By Theorem 6.39 in Chapter 6 it follows that an analytic function can only have a finite number of zeros in a closed bounded set unless it is the constant zero function.) By the Path deformation theorem we can deform C to be a set of small circles C_j each enclosing exactly one zero, z_j, and small circles Γ_k each enclosing exactly one pole, ζ_k, as illustrated in Fig. 5.9. Then

$$\oint_C \frac{f'(z)}{f(z)}\,dz = \sum_j \oint_{C_j} \frac{f'(z)}{f(z)}\,dz + \sum_k \oint_{\Gamma_k} \frac{f'(z)}{f(z)}\,dz.$$

Consider C_j and its interior, where f has a zero at z_j of order n_j. From Definition 3.18 we may write

$$f(z) = (z - z_j)^{n_j} g(z), \qquad \forall z \text{ inside and on } C_j, \tag{5.10}$$

where $g(z)$ is analytic and, assuming C_j is chosen small enough, is not zero inside and on C_j. Using the product rule it is then easy to see that

$$\frac{f'(z)}{f(z)} = \frac{n_j}{z - z_j} + \frac{g'(z)}{g(z)}.$$

Therefore,

$$\frac{1}{2\pi i}\oint_{C_j}\frac{f'(z)}{f(z)}\,dz=\frac{n_j}{2\pi i}\oint_{C_j}\frac{dz}{z-z_j}+\frac{1}{2\pi i}\oint_{C_j}\frac{g'(z)}{g(z)}\,dz=n_j+0,$$

where we have used the Cauchy integral formula on the first term and the Cauchy–Goursat theorem on the second.

Similarly, by Definition 3.26, provided Γ_k is chosen small enough, we have

$$f(z)=\frac{G(z)}{(z-\zeta_k)^{p_k}},\qquad \forall z \text{ inside and on } \Gamma_k, \tag{5.11}$$

where ζ_k is a pole of order p_k and $G(z)$ is analytic and not zero inside and on Γ_k. Differentiating f it is again easy to see that

$$\frac{f'(z)}{f(z)}=-\frac{p_k}{z-\zeta_k}+\frac{G'(z)}{G(z)}.$$

Therefore,

$$\frac{1}{2\pi i}\oint_{\Gamma_k}\frac{f'(z)}{f(z)}\,dz=-\frac{p_k}{2\pi i}\oint_{\Gamma_k}\frac{dz}{z-\zeta_k}+\frac{1}{2\pi i}\oint_{\Gamma_k}\frac{G'(z)}{G(z)}\,dz=-p_k+0,$$

We now sum over j and k to get

$$\frac{1}{2\pi i}\oint_{C}\frac{f'(z)}{f(z)}\,dz=\sum_j n_j-\sum_k p_k=N-P.$$

□

The reason this theorem is called the "Argument" theorem will become more apparent when we prove the Argument principle in Section 5.3.3. Primarily it has to do with the fact that $\log(f(z))$ is a local antiderivative of f'/f. Let $w=f(z)$ and let Γ be the image of C in the w-plane under the mapping f. As C is traversed once, Γ may cross the branch cut for $\log(w)$ multiple times. Nonetheless, the contour Γ can be divided into a number of subcontours each starting and ending on the branch cut. Evaluating $\log(w)$ at each of these points, and noting that the argument of w changes by 2π from one side of the branch cut to the other but the modulus is unchanged, it follows that the integral of w'/w along Γ only depends on the argument of w and ends up being $2\pi i$ times the net number of times the branch cut was crossed.

Example 5.35 Compute $I=\oint_C\frac{f'(z)}{f(z)}\,dz$ where $f(z)=\frac{(z^2-4z+4+2i)^2}{(z^2+2iz-2)^3}$, and C is the circle radius 3 centered at the origin.

By the quadratic formula, the numerator of f is zero when

$$z = 2 \pm \sqrt{4-(4+2i)} = 2 \pm \sqrt{-2i} = 2 \pm (-1+i) = 3-i,\ 1+i.$$

But $3-i$ is outside C, so the only zero inside C is $z = 1+i$, which is a zero of order 2. Applying the quadratic formula to the denominator of f we find

$$z = -i \pm \sqrt{-1+2} = \pm 1 - i.$$

Both these points are inside C and they are poles of order 3. Therefore,

$$I = 2\pi i\,(2-6) = -8\pi i.$$

5.3.2 ROUCHÉ'S THEOREM

Rouché's theorem relates the number of zeros for two functions inside a simple closed contour based on their magnitudes on the contour. Eugène Rouché (1832–1910) proved the theorem that we give below as Corollary 5.38 in the mid-1800s. An improved version of the theorem with a weaker hypothesis was first published by Glicksberg [1976], which we now state and prove.[4]

Theorem 5.36 Improved Rouché's Theorem. *If f and g are analytic inside and on a simple closed contour C and if $|f(z)+g(z)| < |f(z)| + |g(z)|$ on C, then f and g have the same number of zeros inside C (counting order).*

Proof. By triangle inequality we know that

$$|f(z)+g(z)| \le |f(z)| + |g(z)| \qquad \text{for all } z \text{ on } C. \tag{5.12}$$

It is clear that equality holds in (5.12) if and only if one of f or g is a non-negative multiple of the other. The theorem hypothesis is simply that equality *cannot* hold in (5.12). Let $h(z,t) = \frac{f(z)}{g(z)} - t$, for z inside and on C, and $t \in \mathbb{R}$. Since f and g are analytic on C, it follows that the theorem hypothesis implies that $h(z,t)$ is analytic on C and

$$h(z,t) \neq 0, \qquad \forall z \in C,\ 0 \le t.$$

Therefore,

$$I(t) = \frac{1}{2\pi i} \oint_C \frac{\frac{dh(z,t)}{dz}}{h(z,t)}\, dz$$

[4]Irving Glicksberg (1925–) was a mathematics professor at the University of Washington. The author first became aware of this improved version from the text by Matthews and Howell [2012].

is a continuous function of t for $t \geq 0$, and, by the Argument Theorem 5.34, $I(t)$ is an integer: the number of zeros of $h(z,t)$ less the number of poles of $h(z,t)$ inside C. The only way a continuous function of t can be an integer is if it is a constant. But, the numerator of the integrand in the expression for $I(t)$ is actually independent of t and the modulus of the denominator tends to ∞ for large t, so $\lim_{t\to\infty} I(t) = 0$. We conclude that $I(t) \equiv 0$. Further, $h(z,0) = f(z)/g(z)$, hence, $I(0) = N_f - N_g$ where N_f is the number of zeros of f and N_g the number of zeros of g inside C. (Recall that f and g are analytic, and the poles of f/g are therefore the zeros of g.) Since $I(t) \equiv 0$ it follows that $N_f = N_g$. □

Corollary 5.37 *If f and g are analytic inside and on a simple closed contour C and if $|f(z) - g(z)| < |f(z)| + |g(z)|$ on C, then f and g have the same number of zeros inside C (counting order).*

Proof. The corollary hypothesis is equivalent to

$$|f(z) + (-g(z))| < |f(z)| + |-g(z)|\,,$$

thus by Theorem 5.36 f and $-g$ have the same number of zeros inside C. But $-g$ and g must clearly have the same number of zeros. □

Rouché's theorem is usually given in terms of a function F and a perturbation G. If the modulus of the perturbation is not as large as the modulus of F on the curve C the result is that perturbing F by G does not change the number of zeros. We give this formulation of Rouché's theorem in the next corollary.

Corollary 5.38 Rouché's Theorem. *If F and G are analytic inside and on a simple closed contour C and if $|G(z)| < |F(z)|$ on C, then F and $F + G$ have the same number of zeros inside C (counting order).*

Proof. Define $f = F$ and $g = F + G$, so that $G = g - f$. Then

$$|G(z)| < |F(z)| \quad\Longrightarrow\quad |-f(z) + g(z)| < |f(z)| \quad\Longrightarrow\quad |f(z) - g(z)| < |f(z)|\,,$$

and so the result immediately follows from Corollary 5.37. □

Although Corollary 5.38 is clearly a weaker form of the theorem, it is generally adequate for most problems.

Example 5.39 How many roots of $P(z) = 2z^5 - 6z^2 + z + 1 = 0$ lie in the region $1 \leq |z| < 2$?

Let $F(z) = -6z^2$ and $G(z) = 2z^5 + z + 1$ then on $|z| = 1$ we have $|F(z)| = 6$ and $|G(z)| \leq 4$. Therefore, applying the theorem we know that the number of zeros of P inside the circle radius 1 is equal to the number of zeros of F inside this circle, which is 2 (one of order 2).

For the disk $|z| \leq 2$, take $F(z) = 2z^5$ and $G(z) = -6z^2 + z + 1$. Then $|F(z)| = 64$ and $|G(z)| \leq 27$ on $|z| = 2$. From the theorem then, the number of zeros of P inside $|z| = 2$ is equal to the number of zeros of F, which is 5 (one of order 5).

Therefore, P has three zeros in $1 \leq |z| < 2$.

5.3.3 ARGUMENT PRINCIPLE

Consider a simple closed contour C with z_0 in the interior and assume $f(z) \neq 0$ on C. Starting at any point ζ_1 on C set ϕ_1 to be one of the particular values of $\arg(f(\zeta_1))$. Now traverse C in the positive sense to the point ζ on C and allow $\arg(f(z))$ to vary continuously from ϕ_1 to a value ϕ. (The word "continuously" prevents the argument from arbitrarily jumping by a multiple of 2π.) In the limit as ζ approaches ζ_1 in the positive direction, the curve C will have been traversed exactly once in the positive sense. Define ϕ_2 to be this limiting value of $\arg(f(z))$. We define the change in argument for f on C as

$$\Delta_C \arg(f(z)) = \phi_2 - \phi_1.$$

Example 5.40 What is $\Delta_C \arg(f(z))$ for the function $f(z) = iz^2$, where C is any simple closed contour with 0 in the interior? What is the value if 0 is not in the interior?

The phase plot of f is shown in Fig. 2.3b where the phase colors go through two full cycles as the origin is encircled once. Therefore, $\Delta_C \arg(f(z)) = 4\pi$. In contrast, if C does not enclose 0 then the phase colors do not go through a full cycle but return to where they started, hence, the value is 0.

Theorem 5.41 Argument Principle. *Let f be analytic inside and on a simple closed contour C except for a finite number of poles inside C. Also assume that $f(z) \neq 0$ on C. Then*

$$\frac{1}{2\pi}\Delta_C \arg(f(z)) = N - P, \tag{5.13}$$

where N is the number of zeros and P the number of poles of f inside C, respectively, counting order.

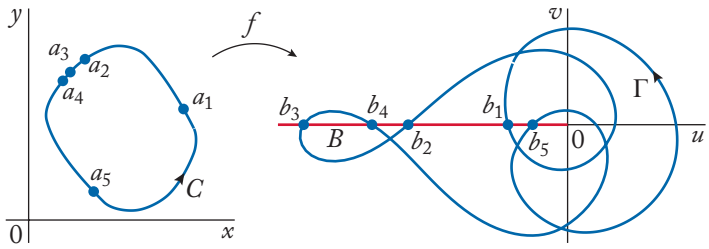


Figure 5.10: The contour C and its image Γ under f. Points a_k on C map to points b_k on Γ where it crosses the branch cut B.

Proof. The contour C satisfies the requirements of the Argument theorem, thus from that theorem we know that

$$N - P = \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz. \tag{5.14}$$

Let $w = f(z)$ and let Γ be the image of C under the mapping f. Although C is a simple closed contour, Γ need not be simple but it must be closed. Also, since f is not zero on C it follows that Γ does not pass through the origin in the w-plane. Define the positive orientation of Γ to be that generated by the positive orientation of C. Select any branch of the function $\log(w)$ with branch cut B extending from $w = 0$ to $w = \infty$. (The principal branch suffices; assume it is selected so B is the negative real axis.) Starting from any point on C, traverse C in the positive sense and label the points a_k on C whose images under f are points on B. Call these corresponding points b_k; see Fig. 5.10. Suppose there are K such points; K may be zero. Then for each section C_k of C from a_k to a_{k+1} (where a_{K+1} is defined to be a_1), it follows that $f(z)$ does not cross the branch cut B and so f'/f is analytic in a domain containing C_k and $\mathrm{Log}(f(z))$ is an antiderivative of f'/f along C_k. Then

$$\int_C \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^{K} \int_{C_k} \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^{K} \mathrm{Log}(f(z))\Big|_{a_k}^{a_{k+1}} = \sum_{k=1}^{K} \mathrm{Log}(b_{k+1}^-) - \mathrm{Log}(b_k^+),$$

where the superscript on the b_k indicates whether the point b_k is on the side of B prior $(-)$ to Γ crossing the branch cut or after $(+)$. Altering the summation index from $k+1$ to k on the first term in the above and using the fact that $b_{K+1} = b_1$ we have

$$\begin{aligned}\int_C \frac{f'(z)}{f(z)}\, dz &= \sum_{k=1}^{K} \mathrm{Log}(b_k^-) - \mathrm{Log}(b_k^+) = \sum_{k=1}^{K} \ln\left|b_k^-\right| - \ln\left|b_k^+\right| + i\left[\arg(b_k^-) - \arg(b_k^+)\right] \\ &= i \sum_{k=1}^{K} \arg(b_k^-) - \arg(b_k^+).\end{aligned}$$

Now if the crossing of B at b_k is in the direction where $\arg(w)$ increases as Γ is traversed in the positive sense, then $\arg(b_k^-) = \pi$ and $\arg(b_k^+) = -\pi$. If the crossing is in the other direction then these values are reversed. (In the example of Fig. 5.10, the crossing at b_3 decreases $\arg(w)$ but the other crossings all increase $\arg(w)$.) Therefore,

$$\int_C \frac{f'(z)}{f(z)}\,dz = i(M^+ - M_-)2\pi,$$

where M^+ is the number of times Γ crosses B with $\arg(w)$ increasing and M^- is the number of times Γ crosses B with $\arg(w)$ decreasing. (If $K = 0$ then then both M^+ and M^- are zero and the integral is zero since $\mathrm{Log}(f(z))$ would be an antiderivative for the integrand on all of C.) But the full change in $\arg(w)$ as Γ is traversed once in the positive sense is clearly $2\pi(M^+ - M^-)$, therefore from (5.14) we have

$$N - P = \frac{1}{2\pi}\Delta_C \arg(f(z)).$$

□

5.3.4 EXERCISES

5.19. Use the Argument theorem to compute $\oint_C \frac{f'(z)}{f(z)}\,dz$ for the following, where C is a circle radius 4.1, centered at the origin.

(a) $f(z) = \dfrac{(z^2+9)^2}{z^3 - 5z^2 + 12z + 18}$, **(b)** $f(z) = \tan(\pi z)$, **(c)** $f(z) = \dfrac{\log(z^2+18)}{\sin z}$,

where for (c), the branch of the logarithm is $\log(z) = \ln|z| + i\arg(z)$, $-\pi < \arg(z) \le \pi$.

5.20. Use the Argument theorem to compute $I = \oint_C \dfrac{(ze^z - 1)/(z+1)^2}{(e^z+1)/(z+1)}\,dz$, where C is the square that bounds the region $|\mathrm{Re}(z)| \le 10$, and $|\mathrm{Im}(z)| \le 10$.

5.21. Find the number of zeros of $4z^3 - 7z^2 + z - 1$ in the annulus $1 \le |z| < 2$.

5.22. Let C be a simple closed contour and let Γ be the image of C under a mapping f. Suppose f is a meromorphic function inside and on C and that $f(z) \ne 0$ on C. If 0 is not in the interior of Γ and there is a ray from 0 to ∞ that does not intersect Γ, then $\Delta_C \arg(f(z)) = 0$.

5.23. Let C be the unit circle $|z| = 1$. For each of the following functions, determine $\Delta_C \arg(f(z))$ and determine the number of times the image of C under f winds around 0 (in the positive direction) as C is traversed once in the positive sense.

(a) $f(z) = z^3$, **(b)** $f(z) = \dfrac{2z - i}{z^2}$, **(c)** $f(z) = \sin(8z)$.

CHAPTER 6

Series

6.1 CONVERGENCE

In this section we define what it means for sequences and series of functions to converge, and provide a number of important results about convergence.

6.1.1 SEQUENCES

Definition 6.1 A sequence of functions on $\mathbb{C}$, $\{u_n(z)\}_{n=1}^{\infty}$, **converges** to the function $U(z)$ if for any $\epsilon > 0$ there exists an $N > 0$ (in general depending on ϵ and z) such that

$$|u_n(z) - U(z)| < \epsilon, \qquad \forall\, n > N.$$

If N is independent of z in a region R then we say that $\{u_n(z)\}_{n=1}^{\infty}$ converges **uniformly** to $U(z)$ in R. R is called the **region of convergence**. If a sequence does not converge, it is said to **diverge**.

The starting point of the sequence is irrelevant in terms of convergence, the counter n could start at any integer. Often we will omit the range of n when labeling a sequence and simply write $\{u_n(z)\}$.

Example 6.2 Where does the series $\{u_n(z)\}_{n=0}^{\infty}$ with $u_n(z) = z^n$ converge? Is the convergence uniform in a region?

We have

$$\lim_{n\to\infty} u_n(z) = \begin{cases} 0 & \text{if } |z| < 1, \\ 1 & \text{if } z = 1, \\ \infty & \text{if } |z| > 1, \\ ? & \text{if } |z| = 1,\ z \neq 1. \end{cases}$$

So $\{u_n(z)\}$ converges to $U(z) \equiv 0$ on $|z| < 1$, converges to $U(z) = 1$ if $z = 1$, and otherwise diverges.

The convergence on $|z| < 1$ is *not* uniform since for $z = re^{i\theta}$, $r < 1$,

$$|u_n(z) - U(z)| = \left|r^n e^{in\theta} - 0\right| = r^n < \epsilon \qquad \Longrightarrow \qquad n > \frac{\ln \epsilon}{\ln r} \to \infty \quad \text{as } r \to 1.$$

Thus, given any ϵ it is impossible to choose N so that the above inequality holds for all $n \geq N$ and all $r < 1$. However, uniform convergence can be obtained by considering a region just slightly smaller, for example, $|r| \leq A$ for some $A < 1$. In this case, given ϵ we can choose $N = \ln \epsilon / \ln A$ and then $n \geq N$ will imply $r^n < \epsilon$.

The above example illustrates the typical way in which sequences that are uniformly convergent behave. Usually, the sequence is convergent on some region but not uniformly convergent on that region. However, if the region is made a finite amount smaller, then uniform convergence can be established.

Given the above definition of convergence, it is simple to show that a sequence will converge if and only of both its real and imaginary parts converge.

Theorem 6.3 *If $u_n(z) = a_n(z) + i\,b_n(z)$ and $U(z) = A(z) + iB(z)$ where a_n, b_n, A, and B are real valued functions of the complex variable z, then $\{u_n(z)\}$ converges (uniformly) to $U(z)$ if and only if $\{a_n(z)\}$ and $\{b_n(z)\}$ converge (uniformly) to $A(z)$ and $B(z)$, respectively.*

Proof. Since the absolute value of both the real and imaginary parts of any complex number are smaller than the modulus of the number we have

$$\left.\begin{array}{l} |a_n(z) - A(z)| \\ |b_n(z) - B(z)| \end{array}\right\} \leq |u_n(z) - U(z)| = \sqrt{(a_n(z) - A(z))^2 + (b_n(z) - B(z))^2}.$$

Considering Definition 6.1, the inequality on the left tells us that if $\{u_n(z)\}$ converges to $U(z)$ then $\{a_n(z)\}$ must converge to $A(z)$ and $\{b_n(z)\}$ must converge to $B(z)$. The equality on the right tells us that if $\{a_n(z)\}$ converges to $A(z)$ and $\{b_n(z)\}$ converges to $B(z)$ then $\{u_n(z)\}$ must converge to $U(z)$. The same holds if uniform convergence is considered. □

The following two results allow one to determine if a sequence converges without having to know precisely to what it converges.

Theorem 6.4 Cauchy Convergence Criterion. *The sequence $\{u_n(z)\}$ converges if and only if given any $\epsilon > 0$ there exists $N > 0$ such that*

$$|u_m(z) - u_n(z)| < \epsilon \qquad \forall\, m, n > N. \tag{6.1}$$

Proof. Suppose $\{u_n(z)\}$ converges to $U(z)$. Thus, for a given $\epsilon > 0$ there exists an integer N such that when $n > N$, we have $|u_n(z) - U(z)| < \epsilon/2$. Then, for m and n both larger than N

we have

$$|u_m(z) - u_n(z)| = |u_m(z) - U(z) - (u_n(z) - U(z))|$$
$$\leq |u_m(z) - U(z)| + |u_n(z) - U(z)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Conversely, if (6.1) holds then there are an infinite number of points $u_n(z)$, $n > N$, that are within the neighborhood $N_\epsilon(u_m(z))$. Then, by the Weierstrass–Bolzano theorem (Theorem 2.1), it follows that the set $\{u_n(z)\}_{n=N+1}^{\infty}$ must have a limit point, call it $U(z)$. Then

$$|u_m(z) - U(z)| = \lim_{n\to\infty} |u_m(z) - u_n(z)| < \epsilon,$$

showing that $\{u_m(z)\}$ converges to $U(z)$. □

Theorem 6.5 Monotone Convergence. *If the real sequence $\{a_n(z)\}$ is monotone nondecreasing (or monotone nonincreasing) and $|a_n(z)| \leq M$ for all $n > 0$, then $\{a_n(z)\}$ converges.*

Proof. Suppose the sequence is monotone nondecreasing and consider the set of all real numbers X such that $a_n(z) \leq X$ for all $n > 0$. Since $X = M$ satisfies this requirement, this set is not empty. Now let A be the infimum of this set, that is, A is the smallest number such that $a_n(z) \leq A$ for all $n > 0$. Necessarily then, for any $\epsilon > 0$ there must be at least one integer $N > 0$ such that $A - a_N(z) < \epsilon$, otherwise $X = A - \epsilon$ is a smaller number than A that is in the set. Further, since the sequence is monotone nondecreasing,

$$|A - a_n(z)| = A - a_n(z) \leq A - a_N(z) < \epsilon \qquad \forall\, n > N.$$

Therefore, $\{a_n(z)\}$ converges to A. The monotone nonincreasing case follows by considering the negative of the sequence. □

6.1.2 SERIES

From a sequence of functions $\{u_n(z)\}_{n=1}^{\infty}$ we define

$$S_n(z) = \sum_{k=1}^{n} u_k(z) = u_1(z) + u_2(z) + \cdots + u_n(z).$$

$S_n(z)$ is called the nth partial sum of the sequence. A series is said to converge if its sequence of partial sums converges; specifically:

Definition 6.6 A series of functions $\sum_{n=1}^{\infty} u_n(z)$ **converges** to the function $U(z)$ if for every $\epsilon > 0$ there exists an integer $N > 0$ such that

$$\left|S_n(z) - U(z)\right| = \left|\sum_{k=1}^{n} u_k(z) - U(z)\right| < \epsilon \qquad \forall\, n > N.$$

If N is independent of z in a region R then we say that $\sum_{n=1}^{\infty} u_n(z)$ converges **uniformly** to $U(z)$ in R. R is called the **region of convergence**. If a series does not converge, it is said to **diverge**.

Although the value of a series depends on all terms, the convergence properties do not depend on any finite number of terms at the start. The sum of the these finite terms is simply a constant that can be added to the series constituted of the remaining terms. As with sequences, we will often omit the specific range of n and simply write $\sum u_n(z)$.

For convergence, since the partial sums must converge, that is, approach the same value, it follows that the terms in the series must approach zero.

Theorem 6.7 *A necessary condition for the series* $\sum u_n(z)$ *to converge is that* $\lim_{n\to\infty} u_n(z) = 0$.

Proof. Assume the series converges to $U(z)$. We have that $u_n(z) = S_n(z) - S_{n-1}(z)$. Taking the limit of both sides of this equation as n tends to infinity, and noting that both $S_n(z)$ and $S_{n-1}(z)$ must converge to $U(z)$ it follows that

$$\lim_{n\to\infty} u_n(z) = \lim_{n\to\infty} S_n(z) - S_{n-1}(z) = U(z) - U(z) = 0.$$

□

Example 6.8 Show that if $u_n(z) = \dfrac{z}{n}$, then $\sum u_z(z)$ satisfies Theorem 6.7 but diverges.

Clearly, the terms u_n satisfy Theorem 6.7, however,

$$S_n(z) = \sum_{k=1}^{n} u_k(z) = \sum_{k=1}^{n} \frac{z}{k} = z \sum_{k=1}^{n} \frac{1}{k}.$$

This last series is the harmonic series which diverges. Thus, we see that the condition in Theorem 6.7 is indeed not sufficient for convergence.

As with sequences, it is easy to show that a complex series converges if and only if both the series of its real parts and the series of its imaginary parts converge.

Theorem 6.9 *If $u_n(z) = a_n(z) + i\,b_n(z)$ and $U(z) = A(z) + iB(z)$ where a_n, b_n, A, and B are real valued functions of the complex variable z, then $\sum u_n(z)$ converges (uniformly) to $U(z)$ if and only if $\sum a_n(z)$ and $\sum b_n(z)$ converge (uniformly) to $A(z)$ and $B(z)$, respectively.*

The proof of this theorem is the same as the proof of Theorem 6.3, with partial sums replacing the sequence terms in that proof.

An important concept for series is absolute convergence.

Definition 6.10 If $\displaystyle\sum_{n=1}^{\infty} |u_n(z)|$ converges then $\displaystyle\sum_{n=1}^{\infty} u_n(z)$ is called **absolutely convergent**.

Theorem 6.11 *Absolute convergence implies convergence.*

Proof. For a series $\sum_{k=1}^{\infty} u_k$, let

$$S_n = \sum_{k=1}^{n} u_k, \qquad T_n = \sum_{k=1}^{n} |u_k|, \qquad \text{and} \qquad A_n = S_n + T_n.$$

Then, since $u_k + |u_k| \geq 0$, it follows that A_n is a monotone nondecreasing sequence. Further,

$$A_n = \sum_{k=1}^{n} u_k + |u_k| \leq 2 \sum_{k=1}^{n} |u_k|,$$

and, since by hypothesis $\sum |u_k|$ converges, it follows that the sequence $\{A_n\}$ is bounded above. By Theorem 6.5 it follows that the sequence $\{A_n\}$ converges. Then

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (A_n - T_n) = \lim_{n\to\infty} A_n - \lim_{n\to\infty} T_n,$$

where the last equality is valid since both limits on the right exist. But this shows that $\lim_{n\to\infty} S_n$ exists, that is, $\sum_{k=1}^{\infty} u_k$ converges. □

The converse of this theorem is not true. It is possible for a series to be convergent without being absolutely convergent.

Definition 6.12 A series that is convergent but not absolutely convergent is called **conditionally convergent**.

Absolutely convergent series may be manipulated without worry of changing the convergence properties.

Theorem 6.13 *The terms of an absolutely convergent series may be re-arranged in any order and all such re-arrangements converge to the same sum. Also, the sum, difference, and product of two absolutely convergent series is absolutely convergent.*

6.1.3 SERIES CONVERGENCE TESTS

The following are a few common convergence test for series. They are provided without proof.

1. **Comparison test**: If $\sum |v_n(z)|$ converges and $|u_n(z)| \leq |v_n(z)|$ then $\sum u_n(z)$ converges absolutely.

   If $\sum |v_n(z)|$ diverges and $|u_n(z)| \geq |v_n(z)|$ then $\sum |u_n(z)|$ diverges, but $\sum u_n(z)$ may or may not converge.

2. **Ratio test**: If $\lim_{n\to\infty} \left|\frac{u_{n+1}(z)}{u_n(z)}\right| = L$, then $\sum u_n(z)$ converges absolutely if $L < 1$ and diverges if $L > 1$. If $L = 1$ this test gives no information.

3. **Integral test**: If $f(x) \geq 0$ for $x \geq a$ then $\sum f(n)$ converges or diverges according as $\lim_{M\to\infty} \int_a^M f(x)\,dx$ converges or diverges.

4. **Alternating series test**: If $a_n \geq 0$, $a_{n+1} \leq a_n$, and $\lim_{n\to\infty} a_n = 0$ then $a_0 - a_1 + a_2 - a_3 + \cdots = \sum_{n=0}^{\infty} (-1)^n a_n$ converges.

5. **Geometric Series**: If $\rho < 1$ then the series $\sum_{n=0}^{\infty} a\rho^n$ converges to $\frac{a}{1-\rho}$.

The ratio test and the geometric series result in particular are two results you should be very familiar with as they will be very useful when we discuss power series, Taylor series, and Laurent series in Sections 6.2–6.4.

Example 6.14 Investigate the convergence of $\sum_{n=0}^{\infty} \frac{1}{z^n}$.

Applying the ratio test we have

$$L = \lim_{n\to\infty} \left|\frac{1/z^{n+1}}{1/z^n}\right| = \lim_{n\to\infty} \frac{1}{|z|}.$$

Therefore, the series diverges for $|z| < 1$ and converges absolutely for $|z| > 1$. The convergence will be uniform on $|z| \geq R$ for any $R > 1$. In the case $|z| = 1$, the nth term does not

go to zero as n increases, hence, the series fails the necessary condition and so must diverge for $|z| = 1$.

Example 6.15 Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

The ratio test does not help in this case since the limit of the ratio of two consecutive terms is $L = 1$. We instead use the integral test. Let $f(x) = 1/x^p$. Then the terms of the series are $f(n)$, $n \geq 1$.

$$\int_1^M f(x)\,dx = \int_1^M \frac{1}{x^p}\,dx = \begin{cases} \left[\frac{1}{1-p}x^{1-p}\right]_1^M = \frac{1}{1-p}\left(M^{1-p} - 1\right) & \text{if } p \neq 1 \\ [\ln(x)]_1^M = \ln M & \text{if } p = 1. \end{cases}$$

In the limit as $M \to \infty$, this gives a finite value only if $p > 1$. In particular, the case $p = 1$ is called the harmonic series, and it diverges.

Example 6.16 Show the following series is conditionally convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

This series converges by the alternating series test, but replacing each term with its modulus results in the harmonic series, which does not converge. So the series is conditionally convergent.

Example 6.17 Investigate the convergence of $\sum_{n=1}^{\infty} \frac{z^n}{n}$.

For $r = |z| < 1$ we have $\left|\frac{z^n}{n}\right| < |z^n| = r^n$, but the geometric series $\sum_{n=1}^{\infty} r^n$ converges for $r < 1$, thus by comparison, the original series must converge absolutely for $|z| < 1$. If $|z| > 1$ then, by L'Hopital's rule,

$$\lim_{n\to\infty} \frac{z^n}{n} = \lim_{n\to\infty} \frac{z^n \log z}{1} = \infty.$$

Hence, the series diverges for $|z| > 1$. Alternatively, we could apply the ratio test to obtain the same results:

$$L = \lim_{n\to\infty} \left|\frac{z^{n+1}/(n+1)}{z^n/n}\right| = \lim_{n\to\infty} \frac{n}{n+1}|z| = |z|.$$

Therefore, by the ratio test the series converges if $|z| < 1$ and diverges if $|z| > 1$.

For $|z| = 1$ it may converge or diverge. For $z = 1$ it clearly diverges, since it is the harmonic series. For $z = -1$ it converges by the alternating series test. Determining whether the series converges or not at other points on the unit circle is more difficult. It turns out that the series is conditionally convergent at all points on the unit circle except $z = 1$.

Example 6.18 Does the series $\sum_{n=1}^{\infty} \frac{\cos(nz)}{n^3}$ converge, and if so for what values of z?

We have

$$\cos(nz) = \frac{e^{inz} + e^{-inz}}{2} = \frac{1}{2}\left[e^{-ny}\left(\cos(nx) + i\sin(nx)\right) + e^{ny}\left(\cos(nx) - i\sin(nx)\right)\right].$$

The series

$$\sum_{n=1}^{\infty} \frac{e^{-ny}\left(\cos(nx) + i\sin(nx)\right)}{2n^3} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{e^{ny}\left(\cos(nx) - i\sin(nx)\right)}{2n^3}$$

cannot converge for $y < 0$ and $y > 0$, respectively, since the nth term does not go to zero. If $y = 0$, then the series converges since for $z = x \in \mathbb{R}$ we have

$$\left|\frac{\cos(nx)}{n^3}\right| \leq \frac{1}{n^3},$$

and $\sum \frac{1}{n^3}$ converges, by Example 6.15. Therefore, the series converges for z on the real axis, but at no other points.

6.1.4 UNIFORM CONVERGENCE RESULTS

Determining whether a series converges uniformly on some region is primarily achieved by the following test.

Theorem 6.19 Weierstrass M-Test. *If $|u_n(z)| \leq M_n$ for all z in a region R (M_n is independent of z), and if $\sum M_n$ converges, then $\sum u_n(z)$ is uniformly convergent in R.*

Example 6.20 Show that $\sum_{n=1}^{\infty} \frac{z^n}{n\sqrt{n+1}}$ is uniformly convergent on $|z| \leq 1$.

We have, when $|z| \leq 1$,

$$\left|\frac{z^n}{n\sqrt{n+1}}\right| \leq \frac{1}{n^{3/2}} = M_n,$$

and by Example 6.15 the series $\sum M_n$ converges. Therefore, by the Weierstrass M-test the original series converges uniformly on $|z| \leq 1$.

Example 6.21 Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + z^2}$ converges uniformly on $|z| \leq 2$.

We have, by triangle inequality,

$$|n^2 + z^2| \geq |n^2| - |z^2| \geq n^2 - 4 \geq \frac{1}{2}n^2, \qquad \text{for } |z| \leq 2 \text{ and } n \geq 3.$$

Therefore, $\left|\dfrac{1}{n^2 + z^2}\right| \leq \dfrac{2}{n^2} = M_n$ for $n \geq 3$. Since the series $\sum M_n$ converges, the original series converges uniformly on $|z| \leq 2$, by the Weierstrass M-test.

There are several important theorems that relate properties of the terms of a uniformly convergent series to properties of the sum to which they converge. We state them without proof.

Theorem 6.22 *If $u_n(z)$ is continuous in a region R, for $n > 0$, and $\sum u_n(z)$ converges uniformly to $S(z)$ in R, then $S(z)$ is continuous in R.*

Thus, uniform convergence maintains continuity. That this fact is not true when the convergence is not uniform is illustrated in the next example.

Example 6.23 Let R be the region consisting of points interior to the unit circle centered at the origin, together with the boundary point $z = 1$. Consider the series $u_1(z) = z$ and $u_k(z) = -z^{k-1} + z^k$, for $k \geq 2$. The nth partial sum of this series is $S_n(z) = z^n$, which converges to 0 if $|z| < 1$, but converges to 1 if $z = 1$. Thus, the function $S(z)$ to which the series converges in R is not continuous.

Now, by the Weierstrass M-test we can show that convergence is uniform on the region $|z| \leq r$, for any $r < 1$, since $|u_n(z)| \leq r^{n-1} + r^n = M_n$ and $\sum M_n$ converges. Therefore, in accord with the theorem, $S(z)$ (in this case identically zero) is continuous on this region.

Theorem 6.24 *If $u_n(z)$ is continuous in a region R for $n > 0$, and $S(z) = \sum u_n(z)$ is uniformly convergent in R then*

$$\int_C \sum u_n(z)\, dz = \sum \int_C u_n(z)\, dz, \qquad \forall C \in R.$$

This theorem tells us that uniformly convergent series can be integrated term by term. It is also true that a uniformly convergent series can be differentiated term by term, which is the result of the next theorem.

Theorem 6.25 *If $u_n(z)$ is analytic in a region R for $n > 0$, and $S(z) = \sum u_n(z)$ is uniformly convergent in R then $S(z)$ is analytic in R and*

$$S'(z) = \frac{d}{dz}\sum u_n(z) = \sum u_n'(z).$$

Example 6.26 The geometric series $\sum_{n=0}^{\infty}(-z)^n$ converges to $\dfrac{1}{1+z}$ for $|z| \leq A$, $A < 1$. Find a series representation for $\dfrac{1}{(1+z)^2}$.

Since $|(-z)^n| \leq A^n$ on $|z| \leq A$, we see that the convergence is uniform on this region. We may therefore differentiate the series term by term yielding

$$\frac{-1}{(1+z)^2} = \sum_{n=0}^{\infty} n(-1)^n z^{n-1}.$$

Hence, dividing by -1 and dropping the first term since it is zero, we have

$$\frac{1}{(1+z)^2} = \sum_{n=1}^{\infty} n(-1)^{n-1} z^{n-1} = \sum_{n=0}^{\infty}(n+1)(-1)^n z^n.$$

Example 6.27 Given that $e^z = \displaystyle\sum_{n=0}^{\infty}\frac{z^n}{n!}$ converges absolutely and uniformly on $|z| \leq A$, for fixed $A \in \mathbb{R}$, find a series representation for $\cos(z)$ and $\sin(z)$ on $|z| \leq A$.

By definition of $\cos(z)$ we have

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}.$$

These two series must converge absolutely on $|z| \leq A$, since $|\pm iz| = |z|$. Therefore, we may re-arrange the terms (adding terms with same value of n in each series) to get

$$\cos(z) = \sum_{n=0}^{\infty}\frac{(1+(-1)^n)i^n z^n}{2n!}.$$

The terms in this series with n odd will be zero and so we can replace n with $n = 2k$ to get

$$\cos(z) = \sum_{k=0}^{\infty} \frac{(i)^{2k} z^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}.$$

This series also converges absolutely and uniformly on $|z| \leq A$. We may do a similar computation to obtain a series for $\sin(z)$, but, alternatively, since the series converges uniformly, we may simply differentiate term by term to get

$$\sin(z) = -\sum_{k=0}^{\infty} \frac{(-1)^k 2k z^{2k-1}}{(2k)!} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(2k-1)!}.$$

6.1.5 EXERCISES

6.1. Let $A \in \mathbb{C}$. Determine a region $R \in \mathbb{C}$ and show from the definition that the sequence

$$u_n = \frac{1}{A + nz} \qquad n \in \mathbb{Z}, \quad n \geq 1,$$

converges to 0 for $z \in R$. That is, for any given $\epsilon > 0$ find an integer N such that $|u_n - 0| < \epsilon$ whenever $n > N$. Is the convergence uniform in R?

6.2. From the definition of convergence for a series show that

$$\sum_{n=0}^{\infty} az^n = \frac{a}{1-z}, \qquad \text{when } |z| < 1,$$

where a is a complex constant. [Hint: The identity $(1-z)(1+z+z^2+\cdots+z^n) = 1 - z^{n+1}$ may be helpful.]

6.3. Find the region of absolute convergence for the following series. Is the region bounded?

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{nz^n e^{n\pi i/4}}{e^n - 1},$ **(b)** $\displaystyle\sum_{n=0}^{\infty} \frac{\pi^{-n}}{n^3 + 4} \left(\frac{z-i}{z+2}\right)^n.$

6.4. Use the Weierstrass M-test to find a region R on which the series $\displaystyle\sum_{k=1}^{\infty} \frac{e^{-k^2 z}}{k^2}$ is uniformly convergent. At what points in $\mathbb{C}$ does the series diverge?

6.5. Let R be the interior and boundary of the square with vertices 0, L, iL, and $L + iL$. Show that the series

$$\sum_{n=0}^{\infty} z^n e^{-nz}$$

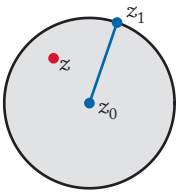


Figure 6.1: A power series centered at z_0 that converges at z_1 also converges at all points z inside the disk of radius $|z_1 - z_0|$.

converges absolutely on R when $L < 1$, but diverges at some points in R if $L > 1$. Does the series converge on R when $L = 1$?

6.2 POWER SERIES

A power series is of the form

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(z - z_0)^n. \tag{6.2}$$

The constant z_0 is called the center of the series. Clearly, the series converges to a_0 at $z = z_0$. Suppose it converges at some other point z_1, then by the ratio test,

$$L_1 = \lim_{n\to\infty} \left| \frac{a_{n+1}(z_1 - z_0)^{n+1}}{a_n(z_1 - z_0)^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |z_1 - z_0| \leq 1.$$

Equality in the above is a possibility since a series may converge when the ratio test gives a value of $L_1 = 1$. Let

$$R = \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|},$$

then the above inequality becomes $|z_1 - z_0| \leq R$. Now consider any point z such that $|z - z_0| < |z_1 - z_0|$ as illustrated in Fig. 6.1. The series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges since by the ratio test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}(z - z_0)^{n+1}}{a_n(z - z_0)^n} \right| = \frac{1}{R} |z - z_0| < \frac{1}{R} |z_1 - z_0| \leq 1.$$

We conclude that if a power series centered at z_0 converges at more than one point, then it converges in a disk of points around z_0.

Definition 6.28 For a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, the largest value of R such that the series converges on the disk $|z-z_0| < R$ is called the **radius of convergence**, and the corresponding disk is called the **region** or **disk of convergence**.

You may have wondered, when you discussed power series in a calculus class, why the interval of convergence for a power series of a real function $f(x)$ centered at x_0 was referred to as a "radius." Now you know. If the function is complexified to $f(z)$, then the complex power series centered at $z_0 = x_0$ converges in a disk of radius R and the interval of the real line inside this disk is $(x_0 - R, x_0 + R)$.

The radius of convergence of a power series may be any nonnegative real number or may be ∞. The series may or may not converge at points on the circle $|z-z_0| = R$. Convergence inside the circle or radius R is absolute, and, if $A < R$, then convergence is uniform on $|z| \leq A$. Since the terms of a power series are obviously analytic, we may therefore integrate and differentiate a power series term by term within its disk of convergence.

Theorem 6.29 Uniqueness of Power Series. *If* $\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} b_n(z-z_0)^n \neq \infty$ *for all* z *in* $|z-z_0| < R$, *then* $a_n = b_n$, $0 \leq n$.

Proof. Since the series are uniformly convergent in $|z-z_0| \leq R_1$ for some $R_1 < R$, we may differentiate them term by term and evaluate at z_0.

$$\begin{aligned} 0 \text{ derivatives evaluated at } z = z_0 &\implies a_0 = b_0 \\ 1 \text{ derivative evaluated at } z = z_0 &\implies a_1 = b_1 \\ &\text{etc.} \end{aligned}$$

□

6.3 TAYLOR SERIES

In this section we will see that a function being analytic on a disk is equivalent to it having a convergent power series, called a Taylor series, on that disk.[1] First we establish the forward direction of this equivalence with Taylor's theorem.

Theorem 6.30 Taylor's Theorem. *Let* $f(z)$ *be analytic inside the disk* $|z-z_0| < R$. *Then at each point in this disk we have*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots .$$

(If $z_0 = 0$ *then the series is sometimes called a Maclaurin series. If* f *is entire then* $R = \infty$.*)*

[1] Brook Taylor (1685–1731) was an English mathematician who is best known for this series that bears his name.

(In the above, the zeroth derivative, $f^{(0)}$, is the function f itself.)

Proof. To start, assume $z_0 = 0$. We will deal with the more general case at the end. Let z be such that $|z| < R$. Let C_0 be a circle of radius $r_0 < R$ centered at 0, but big enough so that z is inside C_0. By Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i}\oint_{C_0} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\oint_{C_0} \frac{f(w)}{w\left(1-\frac{z}{w}\right)}\,dw.$$

Now $(1-c)(1+c+c^2+c^3+\cdots+c^{N-1}) = 1-c^N$ for any $c \in \mathbb{C}$, thus

$$\frac{1}{1-c} = 1+c+c^2+\cdots+c^{N-1}+\frac{c^N}{1-c}, \qquad c \neq 1.$$

Therefore, setting $c = z/w$, the integral may be written as

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C_0} \frac{f(w)}{w}\left[1+\frac{z}{w}+\left(\frac{z}{w}\right)^2+\cdots+\left(\frac{z}{w}\right)^{N-1}+\left(\frac{z}{w}\right)^N \frac{1}{1-\frac{z}{w}}\right] dw \\ &= \frac{1}{2\pi i}\oint_{C_0} \frac{f(w)}{w}+\frac{f(w)z}{w^2}+\frac{f(w)z^2}{w^3}+\cdots+\frac{f(w)z^{N-1}}{w^N}+\frac{f(w)z^N}{w^{N+1}}\left(\frac{1}{1-\frac{z}{w}}\right) dw. \end{aligned}$$

So, by the Cauchy Integral Formulas we have

$$f(z) = f(0)+\frac{f'(0)}{1!}z+\frac{f''(0)}{2!}z^2+\cdots+\frac{f^{(N-1)}(0)}{(N-1)!}z^{N-1}+\rho_N(z),$$

where

$$\rho_N(z) = \frac{z^N}{2\pi i}\oint_{C_0} \frac{f(w)}{w^N(w-z)}\,dw.$$

Now $|w-z| \geq |w|-|z| = r_0-|z|$ for w on C_0. So if $M = \max\limits_{w\in C_0}|f(w)|$, we have

$$|\rho_N(z)| \leq \frac{|z|^N}{2\pi}\frac{M}{r_0^N(r_0-|z|)}2\pi r_0 = \frac{Mr_0}{r_0-|z|}\left(\frac{|z|}{r_0}\right)^N.$$

But $|z| < r_0$, hence, $|\rho_N(z)| \to 0$ as $N \to \infty$. Therefore,

$$f(z) = f(0)+\frac{f'(0)}{1!}z+\frac{f''(0)}{2!}z^2+\cdots = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n, \qquad |z| < R.$$

Now suppose $z_0 \neq 0$. Define a new variable $\zeta = z - z_0$, then define the function g as

$$g(\zeta) := f(\zeta + z_0) = f(z).$$

Clearly, $g(\zeta)$ is analytic when $|\zeta| = |z - z_0| < R$. Therefore, we can apply the above proof to $g(\zeta)$ where the differentiation is with respect to ζ giving

$$g(\zeta) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \zeta^n.$$

But $\frac{dg(\zeta)}{d\zeta} = \frac{df(\zeta+z_0)}{d\zeta} = \frac{df(z)}{dz}\frac{d\zeta}{dz} = \frac{df(z)}{dz}$. The same is true for higher derivatives. Thus, $g^{(n)}(0) = f^{(n)}(z_0)$ and it follows that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \qquad |z - z_0| < R.$$

□

Example 6.31 Find a Taylor series for $f(z) = \log(1 + z)$ around $z = 0$. What is the radius of convergence?

Computing derivatives we get

$$\log(1) = 0, \qquad \frac{d}{dz}\log(1+z)\Big|_{z=0} = \frac{1}{1+z}\Big|_{z=0} = 1,$$

$$\frac{d^2}{dz^2}\log(1+z)\Big|_{z=0} = \frac{-1}{(1+z)^2}\Big|_{z=0} = -1, \quad \frac{d^3}{dz^3}\log(1+z)\Big|_{z=0} = \frac{2}{(1+z)^3}\Big|_{z=0} = 2,$$

$$\frac{d^4}{dz^4}\log(1+z)\Big|_{z=0} = \frac{(-3)(2)}{(1+z)^4}\Big|_{z=0} = -3!, \quad \frac{d^n}{dz^n}\log(1+z)\Big|_{z=0} = (-1)^{n-1}(n-1)!,$$

for $n \geq 1$. So

$$\log(1+z) = 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} z^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots.$$

The radius of convergence is

$$R = \left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\right)^{-1} = \left(\left|\frac{(-1)^n n}{(-1)^{n-1}(n+1)}\right|\right)^{-1} = (1)^{-1} = 1.$$

To which branch does this series converge? The one with $\log(1) = 0$ and with a branch cut emanating from $z = -1$ in such a way that it does not cut the circle radius 1 centered at the origin.

Example 6.32 Find a Taylor series around $z = 0$ for $(1 + z)^p$, where $p \in \mathbb{C}$ is a constant. What is the radius of convergence?

$$\frac{d^n}{dz^n}(1+z)^p\Big|_{z=0} = p(p-1)(p-2)\cdots(p-(n-1)), \qquad n \geq 1.$$

So,

$$(1+z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}z^n + \cdots .$$

If p is an integer, then this series is finite, which means the radius of convergence is ∞. Otherwise,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n!p(p-1)(p-2)\cdots(p-n)}{(n+1)!p(p-1)(p-2)\cdots(p-n+1)}\right| = \lim_{n\to\infty}\left|\frac{p-n}{n+1}\right| = 1,$$

which implies $R = 1$. Again (in the case p is not an integer), the branch to which this series converges is one where $(1+z)^p|_{z=0} = 1$ and where the branch cut is chosen to emanate from $z = -1$ so that it does not cut the unit circle centered at the origin.

The next theorem establishes the reverse direction of the equivalence between analyticity and a convergent Taylor series.

Theorem 6.33 *A power series sums to a function that is analytic at every point inside its circle of convergence.*

Proof. We know that a power series is uniformly convergent inside a closed disk that is interior to the disk of convergence. We know that a uniformly convergent series may be differentiated term by term and the resulting sum of derivatives is the derivative of the original function. Hence, the series sums to an analytic function inside its radius of convergence. □

The above two theorems tell us that being analytic is the same as saying that a function has a Taylor series that converges with some nonzero radius of convergence. The equivalence of analytic functions and convergent Taylor series allows us to determine the radius of convergence for a series in terms of the singularities of f.

Theorem 6.34 *The radius of convergence, R, for a Taylor series of $f(z)$ around the point z_0 is the distance from z_0 to the nearest singularity of f.*

Proof. Let d be the distance from z_0 to the nearest singularity of f. By Theorem 6.33 f is analytic in $|z - z_0| < R$, hence, $R \leq d$. But f has no singularities closer than d from z_0, therefore $f(z)$ is analytic on $|z - z_0| < d$, so that Theorem 6.30 implies $R \geq d$. We conclude that $R = d$. □

This theorem provides a useful way of finding the radius of convergence of a Taylor series for a given function.

Example 6.35 For each of the following, what is the region of convergence for the Taylor series of $f(z)$ around $z = z_0$?

(a) $f(z) = \dfrac{z-4}{(z+2i)(z-3)}$, $z_0 = 1-i$, (b) $f(z) = \dfrac{\cos(z)}{e^{-iz}+1}$, $z_0 = 0$,

(c) $f(z) = \dfrac{1}{1+z^2}$, $z_0 = 0$.

(a) The singularities are $z = -2i$ and $z = 3$. Their distances from z_0 are $|-2i-(1-i)| = |-1-i| = \sqrt{2}$ and $|3-(1-i)| = |2+i| = \sqrt{5}$. We conclude that $R = \sqrt{2}$.

(b) The singularities occur when $e^{-iz}+1 = 0$, that is, when $-iz = \log(-1) = i(\pi + 2\pi k)$, $k \in \mathbb{Z}$. The closest ones to $z_0 = 0$ are $\pm\pi$, thus $R = \pi$.

(c) The singularities are at $z = \pm i$, thus $R = 1$.

Part (c) of the above example illustrates why the Taylor series for the real function $f(x) = \frac{1}{1+x^2}$, which is well-defined and smooth on the entire real line, only has a radius of convergence equal to 1.

Of course not all functions are analytic at all points and so a Taylor series may not exist. For example, the function $f(z) = 1/z$ has no Taylor series centered at $z = 0$. Each branch of a multivalued function f is itself an analytic function, provided the branch cut is not crossed. For any ordinary point of one of the branches of f there will be a Taylor series centered at that point, but its radius of convergence will be no larger than the distance to the branch point.

Students sometimes make the mistake of believing that combinations of polynomials, rational functions, rational roots, exponentials, logarithms, trigonometric and hyperbolic functions, and perhaps a few more special types, describe all possible functions. This is definitely not the case. Even if the functions are restricted to be analytic, the vast majority of such functions cannot be written down in closed form consisting of finite combinations of polynomials, etc. From our above results, we have shown that every analytic function is some convergent Taylor series, and every convergent Taylor series is some analytic function. It is the latter of these two facts that suggests that most analytic functions cannot be written in closed form. For example, start with any closed form function, as complicated as you wish, that is analytic on a disk of radius R centered at z_0 and consider its Taylor series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Now select any sequence of complex numbers $\{\zeta_n\}_{n=0}^{\infty}$, such that $|\zeta_n| \leq 1$, and replace a_n with $\zeta_n a_n$. Then the new series will have terms no larger in modulus than the original terms, and so by the comparison test, the new series must also converge. But this is true for *any* sequence $\{\zeta_n\}_{n=0}^{\infty}$ with elements of modulus no larger than one. In each case we end up with another analytic function. The vast majority of analytic functions can only be represented by a Taylor series.

6.3.1 ZEROS OF ANALYTIC FUNCTIONS

The order of a zero of an analytic function is related to the Taylor series for the function centered at the zero. Recall from Definition 3.18 that we defined a zero of order m of an analytic function f as a point z_0 such that in a region, R, containing z_0 we could write

$$f(z) = (z - z_0)^m g(z), \qquad z \in R,$$

where g was analytic in R and $g(z) \neq 0$ in R. An equivalent definition for a zero of order m for f can be given in terms of a Taylor series.

Definition 6.36 If f is analytic in a region R containing z_0 and if $f^{(n)}(z_0) = 0$ for $n = 0, 1, \ldots, m-1$, and $f^{(m)}(z_0) \neq 0$, then z_0 is a zero of order m of f.

Thus, the definition says that f and its first $m-1$ derivatives are zero at z_0, but the mth derivative is not.

Definition 6.36 is equivalent to Definition 3.18 since under this new definition we have, in a neighborhood of z_0, the following convergent Taylor series for f:

$$\begin{aligned} f(z) &= \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0)^{m+1} + \cdots \\ &= (z - z_0)^m \left[\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0) + \cdots \right]. \end{aligned} \tag{6.3}$$

The terms in square brackets are the Taylor series of some function, call it $g(z)$, which must have the same radius on convergence as the Taylor series for f. (The coefficients of g are just shifted compared to f so the ratio test will yield the same result.) Clearly, g converges to its first term at $z = z_0$, which is nonzero by the definition. Thus, by continuity of g, it must be that $g(z) \neq 0$ for z in a sufficiently small neighborhood of z_0, which we may call R. Hence, we have arrived at the original definition. Conversely, from Definition 3.18, since g is analytic in R and nonzero at z_0, its Taylor series centered at z_0 has a nonzero constant term, $g(z_0)$, and so the Taylor series for $f(z)$ is

$$f(z) = (z - z_0)^m g(z) = (z - z_0)^m \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^n = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^{n+m}.$$

Thus, $f^{(n)}(z_0) = 0$ for $n = 0, 1, \ldots, m-1$, and $f^{(m)}(z_0) = g(z_0) \neq 0$, which is the statement in Definition 6.36.

In Theorem 3.19 we showed that a zero of order m had to be isolated. The same fact follows from Definition 6.36 since sufficiently near z_0 the function f is well approximated by the first term in its Taylor series, which is a constant times $(z - z_0)^m$, whose value cannot be

zero except at z_0. We can now prove our assertion following Theorem 3.19 that if f is zero at z_0 then either f is identically zero, or z_0 is a zero of order m as per Definitions 3.18 and 6.36, in other words, it is isolated.

Theorem 6.37 Isolated Zeros. *If f is analytic at a point z_0 and $f(z_0) = 0$, then either f is identically zero in a neighborhood of z_0 or f is not equal to zero on a deleted neighborhood of z_0.*

Proof. Since f is analytic at z_0 it must be analytic in a neighborhood $N_\delta(z_0)$ for some $\delta > 0$. Therefore, a Taylor series for f centered at z_0 has radius of convergence at least as big as δ. Either this Taylor series has all zero terms, in which case f is identically zero on $N_\delta(z_0)$, or the first nonzero coefficient in the Taylor series is $\frac{f^{(m)}(z_0)}{m!}$ for some $m > 0$ (m cannot be zero since by hypothesis $f(z_0) = 0$). Thus, by Definition 6.36, z_0 is a zero of order m for f. We have already shown above that Definition 6.36 is equivalent to Definition 3.18. Hence, by Theorem 3.19 z_0 must be an isolated zero. □

So unless f is identically zero, zeros of f are isolated. Now you may wonder whether zeros of f are special in this sense. Are other values of f isolated too? The answer is yes.

Corollary 6.38 Isolated Values. *If f is analytic at a point z_0 and $f(z_0) = w_0$, then either f is identically equal to w_0 in a neighborhood of z_0 or $f(z) \neq w_0$ in a deleted neighborhood of z_0.*

Proof. Define $g(z) = f(z) - w_0$. Then $g(z_0) = 0$ and by Theorem 6.37 either g is identically zero on a neighborhood of z_0, which implies f is identically equal to w_0 on that neighborhood, or g is not equal to zero on a deleted neighborhood of z_0, which implies that $f(z) \neq w_0$ in this deleted neighborhood. □

An analytic function, like a polynomial, can only have a finite number of zeros inside a closed bounded region, unless the function is identically zero.

Theorem 6.39 Finite Number of Zeros. *If f is analytic inside and on a simple closed contour C, then either f is identically zero or f has a finite number of zeros inside and on C.*

Proof. Let S be the set of points inside and on C. S is closed and bounded. Suppose f has an infinite number of zeros in S. Then, by the Weierstrass–Bolzano theorem (Theorem 2.1), there must be at least one limit point $z_0 \in S$ for the set of zeros. In other words, there is a sequence of the zeros of f that converges to z_0. By continuity of f, we must have $f(z_0) = 0$ so that z_0 is also a zero of f. If z_0 is a zero of order m, then in a neighborhood $N_\delta(z_0)$ we have $f(z) = (z - z_0)^m g(z)$, with $g(z) \neq 0$ for $z \in N_\delta(z_0)$. But this implies z_0 is the only zero in this

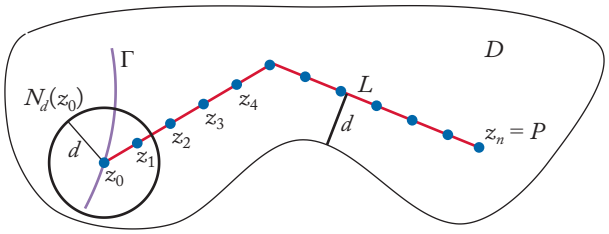


Figure 6.2: Polygonal line L in D joining z_0 on Γ to P, with a sequence of points z_j each a distance less than d from the previous.

neighborhood, which contradicts the fact that there are an infinite number of zeros converging to z_0. Hence, either f is the constant zero function or there must be at most a finite number of zeros inside and on C. □

The fact that zeros of analytic functions are isolated unless the function is identically zero allows us to prove that if two analytic functions in a domain D are identical on some arc in D, then the two functions are identical over all of D. We establish this result with the following lemma and theorem.

Lemma 6.40 *If for each point ζ on an arc Γ, f is analytic at ζ and $f(\zeta) = 0$, then $f(z) \equiv 0$ in any neighborhood $N_\delta(\zeta)$ on which f is analytic.*

Proof. Since f is analytic at ζ, by Theorem 6.37, either f is identically zero in some neighborhood $N_\delta(\zeta)$ of ζ, or f is non-zero in a deleted neighborhood of ζ. But the second alternative is excluded by the lemma hypothesis since f must be zero at points on the arc Γ arbitrarily close to ζ. Therefore, the Taylor series for f centered at ζ has all zero terms, which indicates the radius of convergence is infinity. Thus, the value of δ may be made as large as wished, provided f is analytic on $N_\delta(\zeta)$. □

With the above lemma we can now establish the main result.

Theorem 6.41 Uniqueness of Analytic Functions. *If $f(z)$ and $g(z)$ are analytic on a domain D containing an arc Γ, and if $f(\zeta) = g(\zeta)$ for all ζ on Γ, then $f(z) \equiv g(z)$ on D.*

Proof. Select a point z_0 on Γ and any other point P in D. Since D is connected, there exists a polygonal line L in D going from z_0 to P. Let d be the minimum distance from any point on L to the boundary of D. (If D is all of $\mathbb{C}$ then choose any positive value for d.) Select a sequence of points z_j, $0 \le j \le n$, on L such that $|z_{j+1} - z_j| < d$, $0 \le j < n$, and $z_n = P$. See Fig. 6.2. Define $h(z) = f(z) - g(z)$. Then h is zero on Γ. Further, for any point z_j on L, h is analytic on a neighborhood $N_d(z_j)$, since z_j is at least as far as d from the boundary of D. Since Γ goes

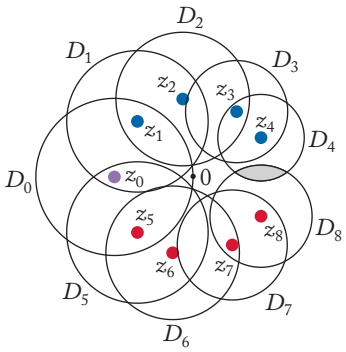


Figure 6.3: Analytic continuation of f by selecting a sequence of points z_{j+1} in $D_j = \{z \in \mathbb{C} : |z - z_j| < R_i\}$ and determining the radius of convergence, R_{j+1}, of a Taylor series for f centered at z_{j+1}.

through z_0 and since h is analytic on $N_d(z_0)$, Lemma 6.40 indicates that h must be identically zero on $N_d(z_0)$. But z_1 is in $N_d(z_0)$, so h is therefore zero along a portion of the line L that goes through z_1. Thus, by Lemma 6.40 we conclude that h is identically zero on $N_d(z_1)$. Continuing in this manner for $j = 1, \ldots, n$ we conclude that h is identically zero on $N_d(z_n)$. Since $P = z_n$ was arbitrarily chosen in D we conclude h is identically zero in D and therefore $f(z) = g(z)$ for all z in D. □

6.3.2 ANALYTIC CONTINUATION

Although a Taylor series for a function f centered at z_0 may have a finite radius of convergence R_0, it may be possible to extend the function beyond the region of convergence so that it remains analytic. The idea is as follows. Define $D_k = \{z \in \mathbb{C} : |z - z_k| < R_k\}$ for $k \geq 0$, then the given Taylor series for f converges on D_0. Select a point z_1 in D_0 and construct a Taylor series for f centered at z_1. Call this series f_1. This new series will converge on a disk D_1 of radius R_1. It is possible that D_1 extends outside the original disk D_0. If it does, then f_1 is called the analytic continuation of f into D_1. This extension, if it exists, is unique, since by Theorem 6.41 there cannot be two functions analytic in D_1 that both take on the values of f in $D_0 \cap D_1$. This process can be repeated, generating a sequence of disks, D_j and functions f_j that are analytic continuations of f; see Fig. 6.3. However, if two such continuation paths eventually intersect, for example, the shaded intersection of D_4 and D_8 in Fig. 6.3, the functions f_4 on D_4 and f_8 on D_8 may or may not be the same, as the next examples illustrate.

Example 6.42 Consider the function $f_0(z) = -\sum_{n=0}^{\infty}(z+1)^n$. This series converges on $D_0 = \{z \in \mathbb{C} : |z + 1| < 1\}$, and from the geometric series result it is clear that the series

converges to $f(z) = 1/z$, which is analytic everywhere except for the simple pole at $z = 0$. Thus, any analytic continuation of f_0 to any point $z \neq 0$ outside D_0 will yield a series that converges to $1/z$, regardless of the continuation path taken.

Example 6.43 In contrast, consider the function $f_0(z) = i\pi - \sum_{n=1}^{\infty} \frac{(z+1)^n}{n}$. This Taylor series convergences on $D_0 = \{z \in \mathbb{C} : |z + 1| < 1\}$. By direct computation, it is not difficult to show that f_0 is the Taylor series centered at $z = -1$ for a portion of a branch of $\log(z)$ such that $\log(-1) = i\pi$, and the branch cut runs from $z = 0$ to ∞ along some ray in the right half-plane, so that it does not intersect D_0. If one were to select a continuation path like D_1 to D_4 in Fig. 6.3, the function f_4 on D_4 would converge to a portion of the branch of $\log(z)$ whose imaginary part would be close to $\theta = 0$. In other words, the analytic function to which all of the series in D_i, $i = 0, 1, 2, 3, 4$, converge is a branch of $\log(z)$ with the branch cut from $z = 0$ to ∞ along some ray going down and to the right so as not to intersect the D_j, $j = 0, 1, 2, 3, 4$. On the other hand, if one were to select a continuation path like D_5 to D_8, the function f_8 on D_8 would converge to a portion of the branch of $\log(z)$ whose imaginary part would be close to $\theta = 2\pi$. The branch cut would need to go up and to the right so as not to intersect any of D_j, $j = 0, 5, 6, 7, 8$. Clearly then, f_4 and f_8 are not the same function; they are on different branches of $\log(z)$.

The idea of analytic continuation is not restricted to disks of convergence for Taylor series. If f_0 is analytic in any domain D_0 and f_1 is analytic on some other domain D_1 that has a nonempty intersection with D_0, and if f_0 and f_1 are equal on this intersection, then f_1 is the analytic continuation of f_0 into D_1.

6.3.3 EXERCISES

6.6. A Taylor series with a finite number of nonzero terms is clearly a polynomial. Show that for any $z_0 \in \mathbb{C}$, the Taylor series for a polynomial of degree n always has a finite number of nonzero terms. What is the highest order nonzero term in this series?

6.7. Given that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

for all $z \in \mathbb{C}$, find Taylor series expansions for the following functions f around the given point z_0. Specify the region of convergence. For (e) and (f) just find the first three

nonzero terms.

(a) $f(z) = e^z,\ z_0 = 1$ **(b)** $f(z) = \cosh z,\ z_0 = 4$

(c) $f(z) = z\sinh z,\ z_0 = 2i$ **(d)** $f(z) = \dfrac{\sin((z-\pi/2)^2)}{z-\pi/2},\ z_0 = \pi/2$

(e) $f(z) = \tan z,\ z_0 = 0$ **(f)** $f(z) = z\coth z,\ z_0 = 0$

6.8. Find the Taylor series centered at 0 for the following functions. Specify the radius of convergence.

(a) $f(z) = e^{z^2+1}$, **(b)** $f(z) = \dfrac{z}{1+z}$, **(c)** $f(z) = z^2 + z$.

6.9. Let A and B be fixed complex numbers, and define $f(z) = \dfrac{A+z}{B+z}$.

(a) Find the Taylor series for $f(z)$ centered at $z = B$.

(b) What is the radius of convergence for the series in (a)?

(c) Compute $\displaystyle\oint_C \frac{f(z)}{(z-B)^5}\,dz$, where C is the circle of radius B centered at B.

(d) Compute $\displaystyle\oint_C \frac{e^z f(z)}{(z-B)^2}\,dz$, where C is the circle of radius B centered at B.

6.10. If the radius of convergence for $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ is R, what is the radius of convergence for the following series?

(a) $\displaystyle\sum_{n=0}^{\infty} n a_n z^n$, **(b)** $\displaystyle\sum_{n=0}^{\infty} a_n^4 z^n$, **(c)** $\displaystyle\sum_{n=0}^{\infty} a_n z^{2n}$.

6.4 LAURENT SERIES

Taylor's theorem tells us that a function that is analytic on a disk has a power series representation on that disk, and vice versa. But what if the function is not analytic on the whole disk but instead is just analytic in an annulus? This is the situation that is addressed by Laurent's theorem.[2]

First recall that Taylor's theorem says that if f is analytic inside $|z - z_0| < R$ then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots.$$

[2]Pierre Alphonse Laurent (1813–1854) was a French mathematician, engineer, and military officer.

By Cauchy's integral formulas we may write these coefficients as

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz, \qquad n \geq 0,$$

where C is any simple closed contour around z_0 contained in $|z - z_0| < R$. Compare this with Laurent's theorem.

Theorem 6.44 Laurent's Theorem. *Let $f(z)$ be analytic in the annular region $R_1 < |z - z_0| < R_2$, then for any point z in this annular region we have*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n} = \sum_{n\in\mathbb{Z}} a_n (z-z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz, \qquad n \in \mathbb{Z},$$

and where C is any simple closed contour in the annular region that has z_0 in its interior.

Proof. Like for the proof of Taylor's theorem, we begin by assuming $z_0 = 0$ and deal with the more general case by a translation of variables at the end. Given z in the annular region $R_1 < |z| < R_2$, choose three circles:

- C_1 centered at 0 radius r_1 such that $R_1 < r_1 < |z|$.
- C_2 centered at 0 radius r_2 such that $|z| < r_2 < R_2$.
- C_3 centered at z radius r_3 small enough so that C_3 lies completely within the annular region and completely within the circle C_2.

These circles are illustrated in Fig. 6.4. The function $\frac{f(w)}{w-z}$ is analytic for w in the annular region, except at the point $w = z$. Thus, by the deformation of path theorem,

$$\oint_{C_2} \frac{f(w)}{w-z}\, dw = \oint_{C_1} \frac{f(w)}{w-z}\, dw + \oint_{C_3} \frac{f(w)}{w-z}\, dw.$$

Now by the Cauchy Integral formula, the last integral is $2\pi i f(z)$, thus

$$f(z) = \frac{1}{2\pi i} \left(\oint_{C_2} \frac{f(w)}{w-z}\, dw - \oint_{C_1} \frac{f(w)}{w-z}\, dw \right). \tag{6.4}$$

Again, recall that $(1-c)(1 + c + c^2 + \cdots + c^{N-1}) = 1 - c^N$ for all $c \in \mathbb{C}$, thus

$$\frac{1}{1-c} = 1 + c + c^2 + \cdots + c^{N-1} + \frac{c^N}{1-c}, \qquad c \neq 1.$$

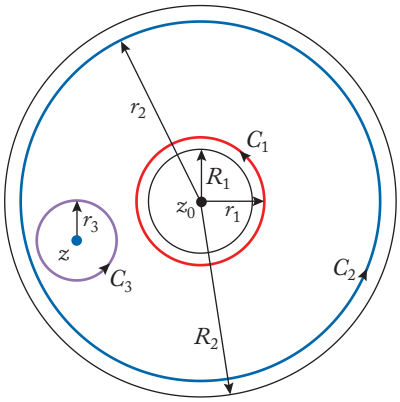


Figure 6.4: Circles used in the proof of Laurent's theorem.

Let $c = z/w$ in the first integrand of (6.4): $\frac{f(w)}{w(1-z/w)}$, and let $c = w/z$ in the second integrand: $\frac{f(w)}{-z(1-w/z)}$, to get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\left[\oint_{C_2} \frac{f(w)}{w}\left(1+\frac{z}{w}+\left(\frac{z}{w}\right)^2+\cdots+\left(\frac{z}{w}\right)^{N-1}+\frac{(z/w)^N}{1-z/w}\right)dw + \right. \\ &\qquad \left. \oint_{C_1} \frac{f(w)}{z}\left(1+\frac{w}{z}+\left(\frac{w}{z}\right)^2+\cdots+\left(\frac{w}{z}\right)^{N-1}+\frac{(w/z)^N}{1-w/z}\right)dw\right] \\ &= \frac{1}{2\pi i}\left[\sum_{n=0}^{N-1} z^n \oint_{C_2} \frac{f(w)}{w^{n+1}}\,dw + z^N \oint_{C_2} \frac{f(w)}{w^N(w-z)}\,dw \right. \\ &\qquad \left. + \sum_{n=1}^{N} z^{-n} \oint_{C_1} \frac{f(w)}{w^{-n+1}}\,dw + z^{-N}\oint_{C_1} \frac{f(w)w^N}{(z-w)}\,dw\right]. \end{aligned}$$

We now show that as $N \to \infty$, the second and fourth term in the above expression are zero. Choose $M > 0$ such that $|f(w)| \le M$ for all $w \in C_1$ and C_2.

$$\left|\frac{z^N}{2\pi i}\oint_{C_2} \frac{f(w)}{w^N(w-z)}\,dw\right| \le \frac{|z|^N M}{2\pi r_2^N(r_2-|z|)}2\pi r_2 = \frac{Mr_2}{r_2-|z|}\left(\frac{|z|}{r_2}\right)^N \to 0,$$

since $|z| < r_2$. Also,

$$\left|\frac{z^{-N}}{2\pi i}\oint_{C_1} \frac{f(w)w^N}{(z-w)}\,dw\right| \le \frac{Mr_1^N}{2\pi\,|z|^N\,(|z|-r_1)}2\pi r_1 = \frac{Mr_1}{|z|-r_1}\left(\frac{r_1}{|z|}\right)^N \to 0,$$

since $r_1 < |z|$. Finally, since $f(w)w^p$ is analytic between C_2 and C and between C_1 and C, the contour integrals may be taken around C. We conclude that

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n, \qquad \text{where} \qquad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz.$$

Now suppose that $z_0 \neq 0$. Define $\zeta = z - z_0$ and the function $g(\zeta) = f(\zeta + z_0) = f(z)$. Then $g(\zeta)$ is analytic when $R_1 < |\zeta| = |z - z_0| < R_2$. We may therefore apply the above proof to $g(\zeta)$ giving

$$g(\zeta) = \sum_{n\in\mathbb{Z}} a_n \zeta^n, \qquad \text{where} \qquad a_n = \frac{1}{2\pi i}\oint_C \frac{g(\zeta)}{\zeta^{n+1}}\,d\zeta.$$

But $d\zeta = dz$ so

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z - z_0)^n, \qquad \text{where} \qquad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz.$$

□

Definition 6.45 For a Laurent series $\sum_{n\in\mathbb{Z}} a_n(z - z_0)^n$, the sum of the nonnegative power terms, $\sum_{n\geq 0} a_n(z - z_0)^n$, is called the **analytic part**, and the sum of the negative power terms, $\sum_{n<0} a_n(z - z_0)^n$, is called the **principal part**.

Just like a function may have many different Taylor series, all centered at different points, there can also be many Laurent series for the same function. In this case however, they may be different because they are centered at different points, or they may be different because they are valid on different annular regions. This is illustrated in the next example. However, if the annular regions of convergence of two Laurent series of the same function overlap, then both series must converge to the same function in that overlap region.

Example 6.46 Find a Laurent series for $f(z) = \dfrac{z}{(z - i)(z + 3)}$

(a) centered at $z = 0$ valid on $3 < |z| < \infty$,
(b) centered at $z = 0$ valid on $1 < |z| < 3$,
(c) centered at $z = 0$ valid on $|z| < 1$,
(d) centered at $z = i$ valid on $0 < |z - i| < R$, for some constant R. How big can R be?

(a) Using partial fractions we may write $\dfrac{1}{(z-i)(z+3)} = \dfrac{A}{z-i} + \dfrac{B}{z+3}$, where $A = \dfrac{1}{3+i}$ and $B = \dfrac{-1}{3+i}$. So

$$f(z) = \frac{Az}{z-i} + \frac{Bz}{z+3} = \frac{A}{1-\frac{i}{z}} + \frac{B}{1+\frac{3}{z}} = A\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n + B\sum_{n=0}^{\infty}\left(\frac{-3}{z}\right)^n,$$

where we have used the fact that the two terms of f are sums of geometric series with $\rho = i/z$ and $\rho = -3/z$, respectively, and $|\rho| < 1$. Thus, the Laurent series is

$$f(z) = \sum_{n=0}^{\infty} \frac{Ai^n + (-3)^n B}{z^n}.$$

(b) In this case, since $1 < |z| < 3$, it is still true that $|i/z| < 1$ and so the term $\dfrac{A}{1-i/z}$ is still the sum of a convergent geometric series. This part of the series is unchanged. However, term $\dfrac{B}{1+3/z}$ is not the sum of a convergent geometric series since $|-3/z| > 1$. So we write this term differently:

$$\frac{Bz}{z+3} = \frac{Bz/3}{1+\frac{z}{3}} = \frac{Bz}{3}\sum_{n=0}^{\infty}\left(\frac{-z}{3}\right)^n,$$

where we have used the geometric series result with $\rho = -z/3$ and $|\rho| < 1$ since $|z| < 3$. So the Laurent series is

$$f(z) = \sum_{n=0}^{\infty}\frac{Ai^n}{z^n} + \sum_{n=1}^{\infty}\frac{B(-1)^{n-1}}{3^n}z^n.$$

(c) For $|z| < 1$, the term for $\dfrac{Bz}{z+3}$ gives the same series as in (b). The term with A must be written as

$$\frac{Az}{z-i} = \frac{Az/i}{1-\frac{z}{i}} = \frac{Az}{i}\sum_{n=0}^{\infty}\left(\frac{z}{i}\right)^n \quad \text{(geometric series with } \rho = z/i \text{ and } |\rho| < 1\text{)},$$

so the Laurent series is

$$f(z) = \sum_{n=1}^{\infty}\left(\frac{A}{i^n} + \frac{(-1)^{n-1}B}{3^n}\right)z^n.$$

In this case the series is actually a Taylor series because the function is analytic on $|z| < 1$.

(d) We write the function in terms of powers of $(z-i)$:

$$f(z) = \frac{(z-i)+i}{(z-i)(z-i+3+i)} = \left(1+\frac{i}{z-i}\right)\left(\frac{1}{z-i+3+i}\right)$$
$$= \left(1+\frac{i}{z-i}\right)\left(\frac{1}{3+i}\right)\left(\frac{1}{1+\frac{z-i}{3+i}}\right).$$

Provided $|z-i| < |3+i| = \sqrt{10}$, the last factor is the sum of a geometric series with $\rho = -\frac{z-i}{3+i}$. Therefore,

$$f(z) = \left(1+\frac{i}{z-i}\right)\left(\frac{1}{3+i}\right)\sum_{n=0}^{\infty}\left(\frac{z-i}{-(3+i)}\right)^n,$$
$$= \frac{1}{3+i}\sum_{n=0}^{\infty}\left(\frac{z-i}{-(3+i)}\right)^n + \frac{i}{3+i}\sum_{n=0}^{\infty}\frac{(z-i)^{n-1}}{(-(3+i))^n}$$
$$= \frac{i/(3+i)}{z-i} + \sum_{n=0}^{\infty}\frac{(-1)^n}{(3+i)^{n+1}}\left(1-\frac{i}{3+i}\right)(z-i)^n,$$

where we obtained the last line by replacing the summation index, n in the second sum to $n+1$, and then collected like terms. From the above, it appears that the largest R can be is $\sqrt{10}$. The series should be convergent until the next singularity of f is encountered, which is at $z=-3$. The distance from $z=-3$ to $z=i$ is $\sqrt{10}$, confirming our expectation that this is the largest possible value for R.

The annular regions in (a), (b), and (c) are mutually exclusive. However, the annular region of (d) overlaps parts of regions in each of (a), (b), and (c). This means, for example, at $z=2$ the series in (b) and the series in (d) both converge to the same value, namely $f(2) = \dfrac{2}{5(2-i)} = \dfrac{2(2+i)}{25}$.

Notice in the above example how the nature of the Laurent series was different in each case. In part (a), the Laurent series had one term in the analytic part and an infinite number of terms in the principal part. For part (b) there were an infinite number of terms in both parts while for part (c) there were no terms in the principal part. Finally, for part (d) there was one term in the principal part and an infinite number of terms in the analytic part. These difference are due to both the location of the center, z_0, and the boundaries of the annulus, R_1 and R_2.

The above example illustrates computation of Laurent series by using the formula for a geometric series. This is a very useful technique. Another useful way of computing a Laurent series is to use known Taylor expansions for some functions, such as e^w and $\cos(w)$.

Example 6.47 Find the Laurent series for $f(z) = \dfrac{e^{z-3}}{z-i}$ centered at $z = i$.

Using the Taylor series for the exponential function we have

$$f(z) = \frac{1}{z-i}\, e^{z-i+i-3} = \frac{e^{i-3}}{z-i}\, e^{z-i} = \frac{e^{i-3}}{z-i} \sum_{n=0}^{\infty} \frac{(z-i)^n}{n!} = e^{i-3} \sum_{n=0}^{\infty} \frac{(z-i)^{n-1}}{n!}.$$

Sometimes long division can be used to compute the first few terms of a Laurent series.

Example 6.48 Find the Laurent series for $f(z) = \dfrac{1}{\cos(z)-1}$ centered at $z = 0$.

$$f(z) = \frac{1}{\cos(z)-1} = \frac{1}{\sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}} = \frac{1}{-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots}.$$

Carrying out long division we obtain

$$\begin{array}{r|l}
 & -\dfrac{2!}{z^2} - \dfrac{(2!)^2}{4!} - \dfrac{3!z^2}{6!} + \cdots \\
\hline
-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots & 1 \\
 & 1 - \dfrac{2!z^2}{4!} + \dfrac{2!z^4}{6!} - \cdots \\
\hline
 & 0 + \dfrac{2!z^2}{4!} - \dfrac{2!z^4}{6!} + \cdots \\
 & \quad + \dfrac{2!z^2}{4!} - \dfrac{(2!)^2 z^4}{(4!)^2} + \cdots \\
\hline
 & 0 + \dfrac{3z^4}{6!} + \cdots
\end{array}$$

Therefore, $f(z) = -\dfrac{2}{z^2} - \dfrac{1}{6} - \dfrac{z^2}{120} + \cdots$.

Just like a Taylor series, the region of convergence for a Laurent series may be expanded until another singularity is encountered. For a Laurent series of f to be valid on an annulus $R_1 < |z - z_0| < R_2$ simply requires that f be analytic in this annulus. Therefore, R_1 can be decreased and R_2 can be increased until a singularity of f is encountered. Conversely, if R_1 can be decreased no further and R_2 can be increased no further without the Laurent series diverging, then there must be a singularity of f on $|z - z_0| = R_1$ and another on $|z - z_0| = R_2$.

A function may not have a convergent Laurent series at some points, for example, it may not have a valid Laurent series centered at a non-isolated singularity. In particular, a branch of

a multivalued function, does *not* have a Laurent series that is valid in a deleted neighborhood of the branch point. This is because a Laurent series requires f to be analytic in an annulus around the center point, and branch points must have a branch cut emanating from them. Thus, the branch cut, which is a set of singularities of the function, must pass through the deleted neighborhood. Now, it is possible to have a Laurent series centered at a branch point, if the inner radius of the annulus of convergence is large enough so that the branch cut terminates at a second branch point in the interior. For example, the function $f(z) = (z(z-1))^{1/2}$ with branch cut on the real axis between 0 and 1, is analytic elsewhere. Hence, by Laurent's theorem must have a Laurent series centered at $z = 0$ and valid on the annulus $1 < |z| < \infty$.

6.4.1 EXERCISES

6.11. Find the Laurent series expansion for $f(z) = 1/(1+z)$ centered at zero and valid on $1 < |z| < \infty$.

6.12. Give two Laurent series expansions centered at 1 for the function $f(z) = 1/(z^2 + 2z - 3)$ and specify the regions in which those expansions are valid.

6.13. Give two different Laurent series expansions centered at i for the function $f(z) = \dfrac{z}{(z-i)^2} + \dfrac{1}{z}$ and specify the regions in which those expansions are valid.

6.14. Let

$$f(z) = \frac{\text{Log}(z+1)}{z(z-5)},$$

where Log is the principal branch of the logarithm.

(a) What is the radius of convergence, R, of a Taylor series for f centered at:

(i) $z = 1 + i$, (ii) $z = 3 - i$, (iii) $z = -2 + i$.

(b) How many different Laurent series centered at isolated singularities of f are possible, where are they centered, and what are their annular regions of convergence?

6.5 ISOLATED SINGULARITIES AGAIN

We gave a set of definitions for isolated singular points in Section 3.4. Now that we have discussed Laurent series, we can give the standard definitions of the three types of isolated singularities.

Let z_0 be an isolated singularity of a function f. Because z_0 is an isolated singularity, by definition f is analytic in a deleted neighborhood of z_0. It follows that f has a Laurent series that converges in this neighborhood. The three types of isolated singularities can be distinguished by the number of terms in the principal part of this Laurent series.

Definition 6.49 If z_0 is an isolated singularity of f, and f has the Laurent series

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(z-z_0)^n, \qquad 0 < |z-z_0| < R, \tag{6.5}$$

then

- z_0 is a **removable singularity** if the principal part of (6.5) is empty, $a_n = 0$ for all $n < 0$,
- z_0 is a **pole of order m** > 0 if the principal part of (6.5) has a positive finite number of nonzero terms with $a_{-m} \neq 0$ and $a_n = 0$ for all $n < -m$; a pole of order one is called a **simple pole**, and
- z_0 is an **essential singularity** if there are an infinite number of nonzero terms in the principal part of (6.5).

For poles of order m the number of nonzero terms in the principal part may be less than m; there may be only one. Similarly for essential singularities it is not necessary for all terms in the principal part to be nonzero, just an infinite number of them.

It is important to emphasize that this definition of isolated singularities uses a Laurent series that is centered at z_0 and that is valid arbitrarily close to z_0, that is, the inner radius of the annulus is 0. Without these restrictions the definition would be useless because we have already seen in Example 6.46 that the number of terms in the principal part may change if either the center of the series is changed or the annulus is changed. The relationship between the number of terms in the principal part of the series and the type of singularity is only valid for Laurent series on a deleted neighborhood of z_0.

Example 6.50 If $f(z) = \frac{1}{z-1}$, then clearly $f(z)$ has a simple pole at $z = 1$. It has a Laurent series centered at $z_0 = 0$ valid on $1 < |z| < \infty$ given by

$$\begin{aligned} f(z) = \frac{1}{z-1} &= \frac{\frac{1}{z}}{1-\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \quad \text{(geometric series with } \rho = \frac{1}{z} \text{ and } |\rho| < 1\text{)}, \\ &= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots . \end{aligned}$$

Clearly, this Laurent series has an infinite number of terms in the principal part.

One might think the reason the above example does not show the relationship between the order of the pole and the number of terms in the principal part of the Laurent series is because the series was not centered at the pole. However, the next example shows that even if we center the series at a pole we still may end up with an infinite number of terms in the principal part.

Example 6.51 If $f(z) = \frac{1}{z(z-2)}$, then clearly $f(z)$ has simple poles at $z = 0$ and $z = 2$. It has a Laurent series centered at $z_0 = 0$ converging on $2 < |z| < \infty$ given by

$$\begin{aligned} f(z) &= \frac{1}{z(z-2)} = \frac{\frac{1}{z^2}}{1-\frac{2}{z}} = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n \quad \text{(geometric series with } |\rho| = \left|\frac{2}{z}\right| < 1\text{)}, \\ &= \frac{1}{z^2} + \frac{2}{z^3} + \frac{2^2}{z^4} + \cdots . \end{aligned}$$

This Laurent series also has an infinite number of terms in the principal part.

Thus, we see that the relationship between the order of a pole z_0 and the number of terms in the principal part of a Laurent series is only valid if the Laurent series is centered at the pole and converges on a deleted neighborhood of z_0, that is, on a region $0 < |z - z_0| < R$.

The fact that the definition for removable singularities and poles given in Definition 6.49 is equivalent to those given in Definitions 3.26 and 3.29 should be clear by simply multiplying the Laurent series (6.5) by $(z - z_0)^m$, $m \geq 0$, and then taking the limit as z approaches z_0, which can be done since the Laurent series is valid in a deleted neighborhood of z_0. Indeed, the nonzero constant A in Definitions 3.26 and 3.29 is the Laurent coefficient a_{-m}.

Example 6.52 By Definition 3.26 it is easy to see that the function $f(z) = e^z/(z-3)^5$ has a pole of order five at $z_0 = 3$ since $\lim_{z\to 3}(z-3)^5 f(z)$ is a nonzero constant. Computing the Laurent series of f we have

$$\begin{aligned} f(z) &= \frac{e^z}{(z-3)^5} = \frac{e^3 e^{z-3}}{(z-3)^5} = e^3\left[\frac{\sum_{n=0}^{\infty}(z-3)^n/n!}{(z-3)^5}\right] = e^3\sum_{n=0}^{\infty}\frac{(z-3)^{n-5}}{n!} \\ &= \frac{e^3}{(z-3)^5} + \frac{e^3}{(z-3)^4} + \frac{e^3}{2!(z-3)^3} + \frac{e^3}{3!(z-3)^2} + \cdots . \end{aligned}$$

The above Laurent series has a finite number of terms in the principal part, the last of which is $e^3/(z-3)^5$, so by the definition given in 6.49, $z_0 = 3$ is a pole of order five.

Example 6.53 Consider the function $f(z) = (\cos z - 1)/z^2$, which has a singularity at $z_0 = 0$. Computing its Laurent series we have

$$\begin{aligned} f(z) &= \frac{\cos(z) - 1}{z^2} = \frac{1}{z^2}\left[\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!} - 1\right] = \frac{1}{z^2}\left[-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right] \\ &= -\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \cdots . \end{aligned}$$

We see then that $f(z)$ is equal to a Taylor series whose radius of convergence is ∞. We conclude from Definition 6.49 that $z_0 = 0$ is a removable singularity. This fact is also seen from

Definition 3.29 by applying L'Hopital's rule twice to compute $\lim_{z\to 0} f(z)$. The singularity is removed by re-defining f to be

$$f(z) = \begin{cases} \frac{\cos z - 1}{z^2} & \text{if } z \neq 0, \\ \frac{1}{2} & \text{if } z = 0. \end{cases}$$

From the definitions of a removable singularity we see that the limit of $f(z)$ is defined as z approaches the singularity. This is related to the following result due to Riemann.

Theorem 6.54 Riemann's Removable Singularity Theorem. *If a function f is bounded and analytic in a deleted neighborhood $N_\delta(z_0) \setminus \{z_0\}$ of a point z_0 then either f is analytic at z_0 or z_0 is a removable singularity of f.*

Proof. Since f is analytic in the deleted neighborhood, it has a Laurent series centered at z_0 with coefficients given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz,$$

where C is a circle radius $\rho < \delta$ centered at z_0. Since f is bounded in $N_\delta(z_0) \setminus \{z_0\}$ we have

$$|f(z)| \leq M, \quad 0 < |z - z_0| < \delta \qquad \Longrightarrow \qquad |a_n| \leq \frac{1}{2\pi} \frac{M}{\rho^{n+1}} 2\pi\rho = M\rho^{-n}.$$

Since ρ can be arbitrarily small, we conclude that $a_n = 0$ for $n < 0$. Thus, the Laurent series for f on $0 < |z - z_0| < \delta$ has no terms in its principal part. If f is defined at z_0 and $f(z_0) = a_0$, then the series is a Taylor series on $|z - z_0| < \delta$, showing that f is analytic at z_0. Otherwise, f may be made analytic by defining $f(z_0) = a_0$, showing that z_0 is a removable singularity of f. □

The definition of essential singularities given in 6.49 immediately allows us to see that the function $f(z) = e^{1/z}$ indeed has an essential singularity at $z = 0$, since, using the Taylor series expansion for e^w with $w = 1/z$ gives

$$f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots.$$

This is a Laurent series for f centered at 0 and valid on $0 < |z| < \infty$.

The behavior of a function near an essential singularity is very weird. Since the modulus of f near a pole of order m grows like $1/|z - z_0|^m$, and since the Laurent series of f at an essential singularity has an infinite number of terms in the analytic part, rather than just m, one might be tempted to say that the modulus of f near an essential singularity grows faster than

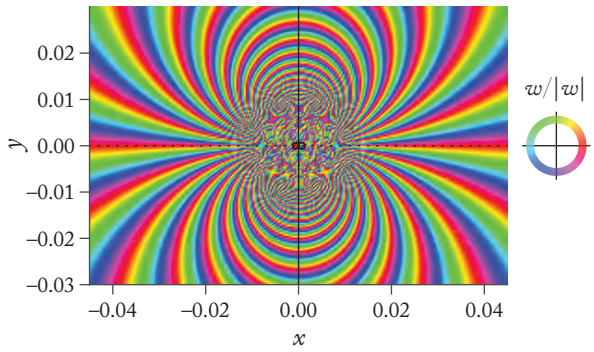


Figure 6.5: Phase plot for $f(z) = e^{1/z}$.

$1/|z - z_0|^M$ for any $M > 0$. But this is not the case. Consider again the function $f(z) = e^{1/z}$. Select any nonzero constant $A \in \mathbb{C}$ and look for solutions of $f(z) = A$. We obtain:

$$f(z) = A \quad \Longrightarrow \quad \frac{1}{z} = \log A = \ln|A| + i\arg(A) = \ln|A| + i(\mathrm{Arg}(A) + 2\pi k), \quad k \in \mathbb{Z}.$$

Since k can take on any integer value we see that $w = 1/z$ has values whose imaginary parts (and therefore also whose moduli) are arbitrarily large in magnitude. This means there are an infinite number of solutions z whose modulus is arbitrarily small. In other words, $f(z) = e^{1/z}$ takes on the value of A an infinite number of times in any deleted neighborhood of 0. A relatively fine-grained phase plot of $e^{1/z}$ is shown in Fig. 6.5 where from the coloring it is evident that the values of $\arg(f(z))$ are very complicated near $z = 0$. That this same behavior is true for any function with an essential singularity is Picard's theorem.[3]

Theorem 6.55 Picard's Theorem. *If z_0 is an essential singularity of f then in each deleted neighborhood of z_0, the function f assumes every value in $\mathbb{C}$, with one possible exception, an infinite number of times.*[4]

The one exception for the case $f(z) = e^{1/z}$ is the value $A = 0$. Although the proof of Picard's theorem is beyond the scope of this text, we will prove a related result.

Theorem 6.56 Weierstrass' Essential Singularity Theorem. *Let z_0 be an essential singularity of a function f and let $A \in \mathbb{C}$ be given. Then for each positive number ϵ, and for each deleted*

[3]Charles Émilie Picard (1856–1941) was a French mathematician, and is known in complex analysis for his "Great" theorem, given here, and his "Little" theorem, which states that every nonconstant entire function takes on all values of $\mathbb{C}$ except perhaps one.

[4]For a proof of this theorem see Titchmarsh [1939, p. 283].

neighborhood $N_\delta(z_0) \setminus \{z_0\}$, *there exists* $z \in N_\delta(z_0) \setminus \{z_0\}$ *such that*

$$|f(z) - A| < \epsilon. \tag{6.6}$$

Proof. Let δ be sufficiently small so that f is analytic in $N_\delta(z_0) \setminus \{z_0\}$. For any given $\epsilon > 0$ suppose there is no z in this deleted neighborhood that satisfies (6.6). Then the function

$$g(z) = \frac{1}{f(z) - A},$$

is bounded (by $1/\epsilon$) and analytic on $N_\delta(z_0) \setminus \{z_0\}$. According to Riemann's removable singularity Theorem 6.54, z_0 must be a removable singularity of g. Thus, we may define g to be analytic at z_0. Then either $g(z_0) \neq 0$, or g has a zero of finite order m at z_0. (The order must be finite, otherwise g would be identically zero, which contradicts its definition.) It follows that the reciprocal

$$\frac{1}{g(z)} = f(z) - A,$$

is either analytic at z_0 or has a pole of order m at z_0, both of which contradict the fact that z_0 is an essential singularity of f. We conclude that there must be a $z \in N_\delta(z_0) \setminus \{z_0\}$ that satisfies (6.6) for the given ϵ. □

6.5.1 EXERCISES

6.15. Suppose $f(z)$ has a pole of order m at z_0. Show that $g(z) = (z - z_0)^m f(z)$ is analytic in a deleted neighborhood of z_0.

6.16. If $f(z)$ and $g(z)$ have poles of order m and n, respectively, at z_0, what type of singularity does each of the following functions have at z_0?

(a) $h(z) = f(z) + g(z)$, **(b)** $h(z) = f(z)g(z)$, **(c)** $h(z) = f(z)/g(z)$.

6.17. Suppose the function $f(z)$ is analytic and has a simple zero at $z = 0$. Suppose that the function $g(z) = f(1/z)$ has a pole of order three at $z = 0$. What can you say about f?

6.18. Consider a function f that has a singularity at $z_1 = 4 + 3i$, and *possibly* a second singularity at some point z_2, but no other singularities. Suppose f has a Taylor series centered at the point $z_0 = 0$, given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

with radius of convergence, R_0. Further, suppose f has a Laurent series centered at z_1 given by

$$f(z) = \sum_{n=-4}^{\infty} b_n (z - z_1)^n,$$

where $b_{-4} \neq 0$, and that this series converges on $0 < |z - z_1| < R_1$.

(a) What is the largest possible value for R_0?

(b) What is the largest possible value for R_1?

(c) What type of singularity is z_1?

For the remainder of this question, assume $a_0 = 3$ and $a_n = \dfrac{5}{n(3i)^n}$, $n \geq 1$.

(d) What is R_0?

(e) What can you say about the existence and location of z_2?

(f) What are the largest and smallest possible values for R_1?

(g) What is $f'(i)$?

(h) Let C be the circle radius one centered at z_0. Compute $\displaystyle\int_C \frac{f(z)}{(z - z_0)^4}\, dz$.

6.19. Show that the function $\sin(1/z)$ has an essential singularity at $z = 0$. For any fixed $\delta > 0$ find all points $z \in N_\delta(0)$ such that $\sin(1/z) = i$.

CHAPTER 7

Residues

Suppose $f(z)$ has an isolated singularity at z_0. Then it has a Laurent series expansion around z_0, that converges on the region $0 < |z - z_0| < R$, for some constant $R > 0$:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n, \qquad \text{where} \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \tag{7.1}$$

and C is a simple closed contour in this region with z_0 in its interior. Since f is analytic in this region, we can deform the path C until it is an arbitrarily small circle around z_0. This would not be possible if the Laurent series was only convergent on the annulus $0 < R_1 < |z - z_0| < R$. The coefficient a_{-1} of the series is given by

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz. \tag{7.2}$$

Only the coefficient a_{-1} involves an integral of f only; all other coefficients involve integrals of f times some power of $(z - z_0)$. Since the contour C can be shrunk to an arbitrarily small circle around z_0, the coefficient a_{-1} is called the residue of f at z_0. (Think of it as the quantity, divided by $2\pi i$, that is left over when you integrate f around z_0 on a circle that becomes vanishingly small.)

Definition 7.1 The coefficient a_{-1} of the Laurent series (7.1) centered at z_0 and valid on $0 < |z - z_0| < R$, with formula given by (7.2), is called the **residue** of f at z_0 and is denoted

$$a_{-1} = \operatorname*{Res}_{z=z_0} f(z).$$

7.1 CALCULATION OF RESIDUES

One could compute a_{-1} directly from the integral formula (7.2) by parameterizing the curve C and doing the contour integral. But there are generally better ways to get the residues of f.

First we consider the case where z_0 is a pole of order m for $f(z)$. In this case, on $0 < |z - z_0| < R$, f has a convergent Laurent series given by

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots.$$

Define

$$g(z) = \begin{cases} (z - z_0)^m f(z) & \text{if } 0 < |z - z_0| < R, \\ a_{-m} & \text{if } z = z_0. \end{cases}$$

Then clearly

$$g(z) = a_{-m} + a_{-m+1}(z - z_0) + \cdots + a_{-1}(z - z_0)^{m-1} + a_0(z - z_0)^m + \cdots,$$

which is a Taylor series for g that must be convergent on $|z - z_0| < R$ since its coefficients are the same as the convergent Laurent series, just shifted. Thus, g is analytic on this region. But by Taylor's theorem we know that the coefficient of $(z - z_0)^{m-1}$ for a Taylor series of $g(z)$ must be the $(m - 1)$-th derivative of g evaluated at z_0 and divided by $(m - 1)!$. Therefore we have established the following formula for the residue at a pole.

Theorem 7.2 Residue at a Pole. *If z_0 is a pole of order m for f, then the residue of f at z_0 is*

$$a_{-1} = \operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \big((z - z_0)^m f(z) \big). \tag{7.3}$$

For poles, the above formula is often very useful for computing residues.

Example 7.3 Find the residue of $f(z) = \dfrac{1}{\sin z}$ at $z_0 = 0$.

The point $z_0 = 0$ is a simple pole, so

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z \to 0} \frac{1}{0!} \left(\frac{z - 0}{\sin z} \right) = \lim_{z \to 0} \frac{1}{\cos z} = 1,$$

where we have used L'Hopital's rule to compute the limit.

Example 7.4 Find the residue of $f(z) = \dfrac{\cos z}{(z - \pi)^3}$ at $z_0 = \pi$.

The point $z_0 = \pi$ is a pole of order three, so we have

$$\operatorname*{Res}_{z=\pi} f(z) = \lim_{z \to \pi} \frac{1}{2!} \frac{d^2}{dz^2} \cos z = \lim_{z \to \pi} \frac{1}{2} (-\cos z) = \frac{1}{2}.$$

Example 7.5 Find the residues of $f(z) = \dfrac{z - 5}{(z - 4i)(z + 3)^2}$ at $z_0 = 4i$ and $z_0 = -3$.

The point $z_0 = 4i$ is a simple pole, so we have

$$\operatorname*{Res}_{z=4i} f(z) = \lim_{z\to 4i} \frac{1}{0!}\left((z-4i)\frac{z-5}{(z-4i)(z+3)^2}\right) = \lim_{z\to 4i}\left(\frac{z-5}{(z+3)^2}\right) = \frac{4i-5}{(4i+3)^2}$$
$$= \frac{(4i-5)(3-4i)^2}{(25)^2} = \frac{(4i-5)(-7-24i)}{(25)^2} = \frac{133+92i}{(25)^2}.$$

The point $z_0 = -3$ is a pole of order two so the residue there is

$$\operatorname*{Res}_{z=-3} f(z) = \lim_{z\to -3} \frac{1}{1!}\frac{d}{dz}\left((z+3)^2\frac{z-5}{(z-4i)(z+3)^2}\right) = \lim_{z\to -3}\frac{d}{dz}\left(\frac{z-5}{z-4i}\right)$$
$$= \lim_{z\to -3}\frac{z-4i-(z-5)}{(z-4i)^2} = \frac{5-4i}{(-3-4i)^2} = \frac{-133-92i}{(25)^2},$$

where we have obtained the last equality by noting that the expression is just the negative of the expression for the residue at $z = 4i$, which we have already computed.

Another very important way of computing residues is to use known series expansions, as the next examples illustrate.

Example 7.6 Find the residue at the essential singularity $z_0 = 0$ of $f(z) = e^{-1/z}$.

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} \qquad \Longrightarrow \qquad e^{-1/z} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,z^n} = 1 - \frac{1}{z} + \frac{1}{2!\,z^2} - \cdots.$$

So the residue is -1.

Example 7.7 Find the residue at $z = 0$ for $f(z) = \dfrac{1}{z(e^z-1)}$.

We can do this in three different ways.

Method 1: The point $z = 0$ is a pole of order 2 since

$$\lim_{z\to 0} z^2 f(z) = \lim_{z\to 0}\frac{z}{e^z-1} = \lim_{z\to 0}\frac{1}{e^z} = 1.$$

Using our formula for the residue at a pole we have

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\left(\frac{z}{e^z-1}\right) = \lim_{z\to 0}\frac{e^z-1-ze^z}{(e^z-1)^2} = \lim_{z\to 0}\frac{e^z-e^z-ze^z}{2(e^z-1)e^z}$$
$$= \lim_{z\to 0}\frac{-z}{2(e^z-1)} = \lim_{z\to 0}\frac{-1}{2e^z} = -\frac{1}{2}.$$

Method 2: Instead of multiple applications of L'Hopital's rule, as in Method 1, we can use the known Taylor series for e^z to compute the residue.

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0}\frac{d}{dz}\left(\frac{z}{e^z-1}\right) = \lim_{z\to 0}\frac{d}{dz}\left(\frac{z}{z+\frac{z^2}{2}+\frac{z^3}{3!}+\cdots}\right)$$
$$= \lim_{z\to 0}\frac{d}{dz}\left(\frac{1}{1+\frac{z}{2}+\frac{z^2}{3!}+\cdots}\right) = \lim_{z\to 0}\frac{-\left(\frac{1}{2}+\frac{z}{3}+\cdots\right)}{\left(1+\frac{z}{2}+\frac{z^2}{3!}+\cdots\right)^2} = -\frac{1}{2}.$$

Method 3: Since $e^z - 1 = z + \dfrac{z^2}{2} + \cdots$ we have $\dfrac{1}{z(e^z-1)} = \dfrac{1}{z^2+\frac{z^3}{2}+\cdots}$. By long division we get

$$\frac{1}{z(e^z-1)} = \frac{1}{z^2} - \frac{1}{2z} + \cdots,$$

hence, the residue is $-1/2$.

7.1.1 EXERCISES

7.1. For each of the following, find the residue of f at z_0.

(a) $f(z) = \dfrac{3z^2}{(z-2)^2}$, $z_0 = 2$ **(b)** $f(z) = z^2\cos\left(\dfrac{1}{z}\right)$, $z_0 = 0$

(c) $f(z) = \dfrac{z}{\sin(z)}$, $z_0 = \pi$ **(d)** $f(z) = \dfrac{1}{\exp\left(\frac{z^2-\pi/2}{\sqrt{3}}\right)+i}$, $z_0 = \dfrac{\sqrt{\pi}}{2}\left(\sqrt{3}-i\right)$

7.2. Give an argument as to why the residue at a removable singularity is always zero.

7.3. Was the fact that the residues calculated in Example 7.5 were negatives of each other just a coincidence? Let z_1 and z_2 be distinct complex numbers and let $A, B \in \mathbb{C}$. Show that if

$$f(z) = \frac{Az+B}{(z-z_1)(z-z_2)^2},$$

then the residue of f at $z = z_1$ is the negative of the residue of f at $z = z_2$. Is the statement true if both A and B are zero?

7.2 THE RESIDUE THEOREM

The main use of Eq. (7.2) is not to compute a residue by a contour integral, but rather the other way around, to have an easy way to compute $\oint_C f(z)\,dz$ by residues. If the residue can

be computed, for example, in any of the ways shown in the previous section, then multiplying that result by $2\pi i$ will give a value for the integral. This is valid for contours enclosing a single singularity. Thanks to path deformation, we can compute integrals around many other contours C using residues of the singularities that are enclosed by C. This is the residue theorem.

Theorem 7.8 Residue Theorem. *Let $f(z)$ be analytic inside and on a simple closed contour C except at a finite number of singularities z_k, $k = 1, 2, \ldots, n$, that occur inside C. Then*

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

Proof. Put small circles C_k of radius R around each point z_k so that none overlap with each other or with C. Since f is analytic between these circles and C, by the Path deformation theorem we get

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz.$$

But, for each z_k, f has a Laurent series that converges on $0 < |z - z_k| < R$ with residue equal to $\dfrac{1}{2\pi i}\displaystyle\oint_{C_k} f(z)\,dz$. Therefore,

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

□

This theorem is just an extension of the formula $\oint_C f(z)\,dz = 2\pi i a_{-1}$, which is valid when C surrounds one singularity of f and where a_{-1} is the residue of the corresponding Laurent series. With the Residue theorem we can deal with contours that surround multiple singularities.

The Residue theorem can also be viewed as a generalization of the Cauchy–Goursat theorem, which would be the special case when there are no singularities of f inside C.

Example 7.9 Compute $\displaystyle\oint_C \frac{\cos(z)}{z \operatorname{Log}\left(1 + \frac{z}{2}\right)}\,dz$ where C is the unit circle centered at the origin and $\operatorname{Log}(w) = \ln|z| + i \operatorname{Arg}(w)$ with $-\pi < \operatorname{Arg}(w) \leq \pi$, is the principal branch of the logarithm.

The branch cut for $\operatorname{Log}\left(1 + \frac{z}{2}\right)$ extends horizontally to the left of $z = -2$ on the real axis, thus the circle C does not intersect it. Hence, the only singularity of the integrand

within C is the point $z = 0$. We have that

$$\begin{aligned} \operatorname{Log}(1+z) &= z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \qquad \Longrightarrow \qquad z \operatorname{Log}\left(1+\frac{z}{2}\right) = \frac{z^2}{2} - \frac{z^3}{8} + \frac{z^4}{24} - \cdots, \\ \cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \end{aligned}$$

So a Laurent series $\sum_{n\in\mathbb{Z}} d_n z^n$ for the integrand satisfies

$$1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \left(\sum_{n\in\mathbb{Z}} d_n z^n\right)\left(\frac{z^2}{2} - \frac{z^3}{8} + \frac{z^4}{24} - \cdots\right).$$

Equating powers of z we get $d_n = 0$ for $n < -2$ and

$$\begin{aligned} z^0: \qquad & 1 = \frac{d_{-2}}{2} && \Longrightarrow \quad d_{-2} = 2, \\ z^1: \qquad & 0 = -\frac{d_{-2}}{8} + \frac{d_{-1}}{2} && \Longrightarrow \quad d_{-1} = \frac{1}{2}. \end{aligned}$$

So the residue of the integrand at $z = 0$ is $1/2$ and

$$\oint_C \frac{\cos(z)}{z \operatorname{Log}\left(1+\frac{z}{2}\right)}\, dz = \pi i.$$

Let us reconsider Example 6.46. In that example we sought Laurent series for $f(z) = \dfrac{z}{(z-i)(z+3)}$ valid on (a) $3 < |z| < \infty$, (b) $1 < |z| < 3$, (c) $|z| < 1$, and (d) $0 < |z-i| < R$. The results were

(a) $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{Ai^n + (-3)^n B}{z^n}$ with $A = \dfrac{1}{3+i}$ and $B = \dfrac{-1}{3+i}$. Thus, the coefficient of z^{-1} was $Ai - 3B$.

(b) $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{Ai^n}{z^n} + \sum_{n=1}^{\infty} \frac{B(-1)^{n+1}}{3^n} z^n$. Thus, the coefficient of z^{-1} was Ai.

(c) $f(z) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{A}{i^n} + \frac{(-1)^n B}{3^n}\right) z^n$. Thus, the coefficient of z^{-1} was 0.

(d) The coefficient of $(z-i)^{-1}$ was $\dfrac{i}{3+i} = Ai$.

Since i and -3 are both simple poles of f it is easy to calculate that the residues of f at these two points are Ai and $-3B$, respectively. We see then immediately that the sum of these two residues was equal to the coefficient of z^{-1} in (a) since the inner circle of the annulus enclosed

both singularities. However, for both (b) and (d), the coefficient of z^{-1} or $(z-i)^{-1}$, respectively, was just the residue at 0, since only that singularity was enclosed by the inner circle of the annulus in those problems. Finally, the coefficient of z^{-1} was zero in (c) since no singularities were inside the region.

7.2.1 EXERCISES

7.4. Use the Residue theorem to compute $\oint_C f(z)\,dz$ where C is the circle of radius 5 centered at the origin.

$$\textbf{(a)}\ z^3 e^{1/z}, \qquad \textbf{(b)}\ \frac{z^2-4}{z^2-4z+16}, \qquad \textbf{(c)}\ \frac{z}{2-\sin z}.$$

[For (c) be careful to find all the singularities of the function. You will need a calculator to verify which ones are inside C.]

7.5. Let Log be the principal branch of the logarithm. Evaluate

$$I = \oint_C \frac{\operatorname{Log}(5-z)}{(z-i)^2} + \frac{1}{e^z+i}\,dz,$$

(a) if C is the circle radius 3 centered at $2i$,

(b) if C is the circle radius 2 centered at 2.

7.6. Suppose f has a Laurent series given by

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n, \qquad R < |z| < \infty, \tag{7.4}$$

where $R > 0$. Let C be a simple closed contour completely within the region $R < |z| < \infty$, which has zero in its interior. We know that

$$\oint_C f(z)\,dz = 2\pi i a_{-1},$$

and that a_{-1} is the sum of the residues of f inside C. Thus, to compute $\oint_C f(z)\,dz$ one could compute all the residues of f inside C. Part (a) of this problem illustrates an alternative way to compute $\oint_C f(z)\,dz$, namely by computing a *single* residue of a related function. It works because the series (7.4) is valid out to ∞, that is, when f has no singularities outside C.

(a) Replace z with $1/z$ in the expressions in (7.4) to find a Laurent series for $f(1/z)$ and its region of validity. Now show that

$$\oint_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f(1/z)\right). \tag{7.5}$$

(b) Use the result (7.5) to compute $\oint_C f(z)\,dz$ where $f(z) = z^{11}(z-1)/(z^{12}+64)$ and C is the circle or radius two centered at the origin.

(c) Use the result (7.5) to prove the following theorem.

Suppose $f(z) = P(z)/Q(z)$ where P and Q are polynomials and the degree of Q is two or more larger than the degree of P. Let C be a simple closed contour that completely encloses all the zeros of $Q(z)$. Then $\oint_C f(z)\,dz = 0$.

(d) Verify the result of (c) for the function $f(z) = (z-1+5i)/(z^3+iz^2+6z)$ by finding all the residues and showing that they add to zero.

7.3 CALCULATION OF CERTAIN REAL INTEGRALS

In this section we illustrate how certain real integrals can be computed by complexifying the problem in an appropriate way and then applying the residue theorem.

7.3.1 INTEGRALS OF THE FORM $\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$

The technique for this type of integral is to define $z = e^{i\theta}$ and use the fact that $\cos\theta$ and $\sin\theta$ may be expressed as

$$\cos\theta = \frac{z+z^{-1}}{2}, \qquad \sin\theta = \frac{z-z^{-1}}{2i}.$$

Further, we have that $dz = ie^{i\theta}d\theta = iz\,d\theta$. It follows that

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta = \oint_C F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{1}{iz}\,dz,$$

where C is the unit circle centered at the origin. The Residue theorem may then be applied to this last integral to compute its value.

Example 7.10 Evaluate $I = \displaystyle\int_0^{2\pi} \frac{1+\sin\theta}{2+\cos\theta}\,d\theta$.

Using the above transformations we have

$$I = \oint_C \frac{1+\frac{z-z^{-1}}{2i}}{2+\frac{z+z^{-1}}{2}}\frac{1}{iz}\,dz = \oint_C \frac{z^2+2iz-1}{-z\,(z^2+4z+1)}\,dz.$$

Let $f(z)$ be the integrand. The denominator of f is zero at $z = 0$ and $z = -2 \pm \sqrt{3}$. Of these only $z_1 = 0$ and $z_2 = -2+\sqrt{3}$ are inside the unit circle C centered at 0, and they

are simple poles. Then, by the Residue theorem,

$$
\begin{aligned}
I &= 2\pi i \sum_{k=1}^{2} \operatorname*{Res}_{z=z_k} f(z) = 2\pi i \left(\lim_{z\to 0} (z-z_1) f(z) + \lim_{z\to z_2} (z-z_2) f(z) \right) \\
&= 2\pi i \left(\frac{-1}{-1} + \lim_{z\to -2+\sqrt{3}} \frac{z^2+2iz-1}{-z(z-(-2-\sqrt{3}))} \right) \\
&= 2\pi i \left(1 + \frac{4-4\sqrt{3}+3-4i+2\sqrt{3}i-1}{(2-\sqrt{3})(2\sqrt{3})} \right) \\
&= \frac{2\pi i}{(2-\sqrt{3})2\sqrt{3}} \left((2-\sqrt{3})(2\sqrt{3}) + 6 - 4\sqrt{3} + i(-4+2\sqrt{3}) \right) \\
&= \frac{2\pi i}{(2-\sqrt{3})2\sqrt{3}} \left(-2i(2-\sqrt{3}) \right) = \frac{2\pi}{\sqrt{3}}.
\end{aligned}
$$

7.3.2 IMPROPER REAL INTEGRALS

In real analysis there are two types of improper integrals:

1. At least one of the limits of integration is unbounded; for example, $\int_0^\infty \sin(x)\,dx$ or $\int_{-\infty}^{\infty} x\,dx$.
2. The integrand is unbounded; for example, $\int_0^1 \frac{1}{x}\,dx$ or $\int_{-\pi}^{\pi} \frac{1}{1-\cos(x)}\,dx$.

Be definition, these improper integrals are said to converge if the "improperness" is removed, but in the limit the integral is bounded. So, if $f(x)$ is defined for all $x \in \mathbb{R}$, and if $g(x)$ is unbounded as $x \to 0$, then

$$\int_0^\infty f(x)\,dx \quad \text{converges to} \quad \lim_{R\to\infty} \int_0^R f(x)\,dx \quad \text{if the limit exists,}$$

$$\int_{-\infty}^{\infty} f(x)\,dx \quad \text{converges to} \quad \lim_{R\to-\infty} \int_R^0 f(x)\,dx + \lim_{R\to\infty} \int_0^R f(x)\,dx \text{ if } \textit{both} \text{ limits exist,}$$

$$\int_0^1 g(x)\,dx \quad \text{converges to} \quad \lim_{\epsilon\to 0} \int_\epsilon^1 g(x)\,dx \quad \text{if the limit exists,}$$

$$\int_{-1}^1 g(x)\,dx \quad \text{converges to} \quad \lim_{\epsilon\to 0^-} \int_{-1}^{\epsilon} g(x)\,dx + \lim_{\epsilon\to 0^+} \int_\epsilon^1 g(x)\,dx \quad \text{if } \textit{both} \text{ limits exist.}$$

Note that in this definition, when the improperness is "two-sided" it is required that two limits approaching from either side are both finite. There is a slightly weaker definition of convergence for these "two-sided" improper integrals. called the Cauchy Principal Value (P.V.). It is defined as

$$P.V.\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx,$$
$$P.V.\int_{-1}^{1} g(x)\,dx = \lim_{\epsilon\to 0^+}\left[\int_{-1}^{-\epsilon} g(x)\,dx + \int_{\epsilon}^{1} g(x)\,dx\right].$$

This is indeed a weaker definition since for $f(x) = x$ we have

$$\int_{-\infty}^{\infty} x\,dx = \lim_{R\to-\infty}\int_{R}^{0} x\,dx + \lim_{R\to\infty}\int_{0}^{R} x\,dx \quad \text{does not exist since neither limit exists,}$$

however,

$$P.V.\int_{\infty}^{\infty} x\,dx = \lim_{R\to\infty}\int_{-R}^{R} x\,dx = \lim_{R\to\infty}\left[\frac{x^2}{2}\right]_{x=-R}^{x=R} = \lim_{R\to\infty}\left[\frac{R^2}{2} - \frac{R^2}{2}\right] = 0.$$

Similarly, the stronger definition of the improper integral $\displaystyle\int_{-1}^{1}\frac{1}{x}\,dx$ does not exist because neither of the limits

$$\lim_{\epsilon\to 0^-}\int_{-1}^{\epsilon}\frac{1}{x}\,dx \qquad \text{and} \qquad \lim_{\epsilon\to 0^+}\int_{\epsilon}^{1}\frac{1}{x}\,dx \qquad \text{exists.}$$

However,

$$P.V.\int_{-1}^{1}\frac{1}{x},dx = \lim_{\epsilon\to 0^+}\left[\int_{-1}^{-\epsilon}\frac{1}{x}\,dx + \int_{\epsilon}^{1}\frac{1}{x}\,dx\right]$$
$$= \lim_{\epsilon\to 0^+}\left[\ln|-\epsilon| - \ln|-1| + \ln|1| - \ln|\epsilon|\right] = 0.$$

The Residue theorem can be used to evaluate the Cauchy Principal Value of certain types of improper integrals. The general idea is to complexify the problem by replacing x with z, then define a contour C on the complex plane that includes, in an appropriate limit, the portion of the real axis in the original problem. If one can then calculate the integral on the remaining portion of C, one can use this value and the Residue theorem, which gives the value of the integral around all of C, in order to find requested integral. We shall dispense with the "P.V." notation in front of the integral sign, and instead simply assume that the improper integral is to be computed as a Cauchy Principal Value.

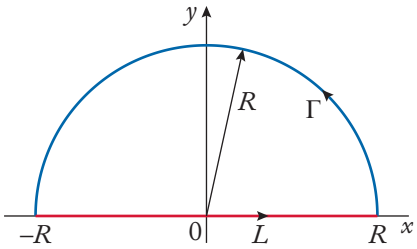


Figure 7.1: The contour C composed of the straight line L and the semicircle Γ.

Integrals of the form $\int_{-\infty}^{\infty} F(x)\,dx$

Let C be the contour made up of the straight line L from $-R$ to R on the real axis and the half circle Γ of radius R centered at 0 and lying in the upper half-plane, as shown in Fig. 7.1. Then

$$\oint_C F(z)\,dz = \int_L F(x)\,dx + \int_\Gamma F(z)\,dz.$$

The integral around C may be computed using the Residue theorem. If we can show that in the limit as R tends to infinity, the integral along Γ is zero, then the integral around C will be equal to the Cauchy Principal Value of $\int_{-\infty}^{\infty} F(x)\,dx$.

Whether the integral along Γ will be zero or not depends on the function F that is being integrated. The next theorem gives a class of functions for which the assertion is true.

Theorem 7.11 *Let Γ be the contour $z(\theta) = Re^{i\theta}$, $0 \le \theta \le \pi$. If when R is sufficiently large, $|F(z)| \le \dfrac{M}{R^k}$ for $z \in \Gamma$, where $k > 1$ and M are constants, then*

$$\lim_{R\to\infty} \int_\Gamma F(z)\,dz = 0.$$

Proof. Bounding the modulus of the integral we have

$$\lim_{R\to\infty} \left| \int_\Gamma F(z)\,dz \right| \le \lim_{R\to\infty} \frac{M}{R^k}\pi R = \lim_{R\to\infty} \frac{M\pi}{R^{k-1}} = 0,$$

since $k > 1$. Therefore, $\lim_{R\to\infty} \int_\Gamma F(z)\,dz = 0$. □

Corollary 7.12 *If* $F(z) = \dfrac{P(z)}{Q(z)}$, *where* P *and* Q *are polynomials with* $\deg Q \geq 2 + \deg P$ *then*

$$\lim_{R\to\infty} \int_\Gamma F(z)\,dz = 0.$$

Proof. Let p and q be the degrees of P and Q, respectively, then, as $|z| = R$ gets large, the leading terms of P and Q dominate so that on Γ,

$$|F(z)| = \left|\frac{P(z)}{Q(z)}\right| \approx \left|\frac{a_p z^p}{b_q z^q}\right| = \frac{|a_p|}{|b_q|\,R^{q-p}}.$$

Since $q - p \geq 2$, it follows that $|F| \leq \frac{M}{R^k}$ where $M = \frac{|a_p|}{|b_q|}$ and $k > 1$. Theorem 7.11 then applies. □

Example 7.13 Evaluate $I = \displaystyle\int_{-\infty}^{\infty} \frac{a}{x^4 + b}\,dx$, for $b > 0$.

Let C be the path $[-R, R]$ on the real axis together with the arc Γ defined by $z(\theta) = Re^{i\theta}$, $0 \leq \theta \leq \pi$.

Let $F(z) = \dfrac{a}{z^4 + b}$. Then by Corollary 7.12 $\lim\limits_{R\to\infty} \displaystyle\int_\Gamma F(z)\,dz = 0$. Therefore,

$$\lim_{R\to\infty} \oint_C F(z)\,dz = \lim_{R\to\infty} \left[\int_{-R}^{R} F(x)\,dx + \int_\Gamma F(z)\,dz\right] = \int_{\infty}^{\infty} F(x)\,dx + 0 = I.$$

Now the integral on the left encloses all of the upper half-plane. The only singularities of F are $z = (-b)^{1/4} = |b|^{1/4} e^{i\left(\frac{\pi + 2k\pi}{4}\right)}$, $k = 0, 1, 2, 3$. Two of these are in the upper half-plane, namely $z_1 = |b|^{1/4} e^{i\pi/4}$ and $z_2 = |b|^{1/4} e^{i3\pi/4}$. So by the Residue theorem

$$\lim_{R\to\infty} \int_C F(z)\,dz = 2\pi i \left(\operatorname*{Res}_{z=z_1} F(z) + \operatorname*{Res}_{z=z_2} F(z)\right).$$

Both singularities are simple poles so, using L'Hopital's rule we have

$$\operatorname*{Res}_{z=z_1} F(z) = \lim_{z\to z_1} (z - z_1)\frac{a}{z^4 + b} = \lim_{z\to z_1} \frac{a}{4z^3} = \frac{a}{4b^{3/4} e^{i3\pi/4}}.$$

Similarly, $\operatorname*{Res}\limits_{z=z_2} F(z) = \dfrac{a}{4b^{3/4} e^{i9\pi/4}}$. Summing these two residues gives

$$\frac{a}{4b^{3/4}}\left(\frac{\sqrt{2}}{-1+i} + \frac{\sqrt{2}}{1+i}\right) = \frac{a\sqrt{2}}{4b^{3/4}}\left(\frac{-1-i}{2} + \frac{1-i}{2}\right) = \frac{-ai\sqrt{2}}{4b^{3/4}}.$$

Therefore, $I = 2\pi i\left(\dfrac{-ai\sqrt{2}}{4b^{3/4}}\right) = \dfrac{a\pi\sqrt{2}}{2b^{3/4}}$.

Example 7.14 Evaluate $I = \displaystyle\int_0^\infty \frac{x^2}{(x^2+3)^2}\,dx$.

Since the integrand is even, $2I$ is the integral from $-\infty$ to ∞. Let C be the path as in the previous example. Let $F(z) = \dfrac{z^2}{(z^2+3)^2}$, then by Corollary 7.12 $\lim\limits_{R\to\infty}\displaystyle\int_\Gamma F(z)\,dz = 0$. Therefore,

$$\lim_{R\to\infty}\oint_C F(z)\,dz = \lim_{R\to\infty}\left[\int_{-R}^{R} F(x)\,dx + \int_\Gamma F(z)\,dz\right] = \int_{-\infty}^{\infty} F(x)\,dx + 0 = 2I.$$

The poles of F are $z = \pm\sqrt{3}i$. Both are order two, but only $z = \sqrt{3}i$ is in the interior of C. The residue of F at this pole is

$$\begin{aligned}\operatorname*{Res}_{z=\sqrt{3}i} F(z) &= \lim_{z\to\sqrt{3}i}\frac{d}{dz}\left((z-\sqrt{3})^2F(z)\right) = \lim_{z\to\sqrt{3}i}\frac{d}{dz}\left(\frac{z^2}{(z+\sqrt{3}i)^2}\right)\\ &= \lim_{z\to\sqrt{3}i}\frac{2z(z+\sqrt{3}i)^2 - z^2 2(z+\sqrt{3}i)}{(z+\sqrt{3}i)^4} = \lim_{z\to\sqrt{3}i}\frac{2z(z+\sqrt{3}i) - 2z^2)}{(z+\sqrt{3}i)^3}\\ &= \frac{2(\sqrt{3}i)^2}{(2\sqrt{3}i)^3} = \frac{1}{4\sqrt{3}i} = \frac{-i}{4\sqrt{3}}.\end{aligned}$$

Thus,

$$I = \frac{1}{2}\left(2\pi i\left(\frac{-i}{4\sqrt{3}}\right)\right) = \frac{\pi}{4\sqrt{3}}.$$

Integrals of the form $\int_{-\infty}^{\infty} F(x)\cos(mx)\,dx$ and $\int_{-\infty}^{\infty} F(x)\sin(mx)\,dx$

Integrals with integrands containing a factor of $\sin(mx)$ or $\cos(mx)$ can also be evaluated in a similar manner. The key idea is that if L is the real axis from $-R$ to R we may write

$$\int_{-R}^{R} F(x)\cos(mx)\,dx = \operatorname{Re}\left(\int_L F(z)e^{imz}\,dz\right)$$

and

$$\int_{-R}^{R} F(x)\sin(mx)\,dx = \operatorname{Im}\left(\int_L F(z)e^{imz}\,dz\right).$$

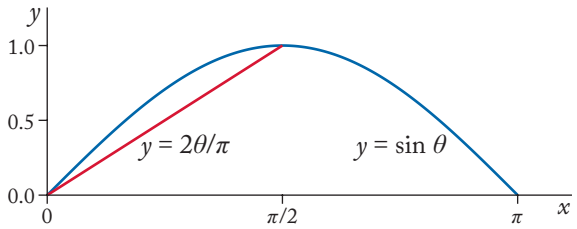


Figure 7.2: The straight line $y = 2\theta/\pi$ is below $y = \sin\theta$ for $0 \le \theta \le \pi/2$.

Thus, we consider the contour C made up of L and Γ and compute

$$\int_C F(z)e^{imz}\,dz = \int_L F(z)e^{imz}\,dz + \int_\Gamma F(z)e^{imz}\,dz$$

by the Residue theorem. Then, as before, if we can show that the integral along Γ goes to zero as R goes to infinity, then either the real or imaginary part of the integral around C will be the Principal Value of the original integral.

First we establish the conditions under which the integral of $F(z)e^{mz}$ along Γ vanishes as R gets large.

Theorem 7.15 *Let Γ be the contour $z(\theta) = Re^{i\theta}$, $0 \le \theta \le \pi$. If when R is sufficiently large, $|F(z)| \le \dfrac{M}{R^k}$ for $z \in \Gamma$, where $k > 0$ and M are constants, then*

$$\lim_{R\to\infty} \int_\Gamma F(z)e^{imz}\,dz = 0, \qquad m \in \mathbb{R},\ m > 0.$$

Note that in this theorem the condition on k is more relaxed than in Theorem 7.11. The presence of the exponential term is what allows us to do this, however the proof is more involved than the previous theorem.

Proof. Parameterizing the contour and then using the usual inequalities of integrals we get

$$\left|\int_\Gamma F(z)e^{imz}\,dz\right| = \left|\int_0^\pi F(Re^{i\theta})e^{im(R\cos\theta + iR\sin\theta)}Rie^{i\theta}\,d\theta\right| \le \int_0^\pi \left|F(Re^{i\theta})\right| Re^{-mR\sin\theta}\,d\theta$$
$$\le \frac{M}{R^k}R\int_0^\pi e^{-mR\sin\theta}\,d\theta = \frac{2M}{R^k}R\int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta,$$

where the last inequality is due to the fact that $\sin\theta$ is even around $\theta = \pi/2$. Now, as shown in Fig. 7.2, the line $y = 2\theta/\pi$ is underneath $\sin\theta$ for $0 \le \theta \le \pi/2$.

Therefore,

$$\left|\int_\Gamma F(z)e^{imz}\,dz\right| \le \frac{2M}{R^k}R\int_0^{\pi/2} e^{-mR2\theta/\pi}\,d\theta = \frac{2M}{R^k}R\left[\frac{-\pi}{2mR}e^{-2mR\theta/\pi}\right]_0^{\pi/2}$$
$$= \frac{M\pi}{mR^k}\left(-e^{-mR}+1\right).$$

Since $k > 0$, the above quantity goes to zero in the limit as R goes to infinity. □

Corollary 7.16 *If* $F(z) = \dfrac{P(z)}{Q(z)}$, *where* P *and* Q *are polynomials with* $\deg Q \ge 1 + \deg P$ *then*

$$\lim_{R\to\infty}\int_\Gamma F(z)e^{imz}\,dz = 0, \qquad \text{for } m > 0.$$

Proof. Let p and q be the degrees of P and Q, respectively. Then, as $|z| = R$ gets large, the leading terms of P and Q dominate so that on Γ,

$$|F(z)| = \left|\frac{P(z)}{Q(z)}\right| \approx \left|\frac{a_p z^p}{b_q z^q}\right| = \frac{|a_p|}{|b_q|\,R^{q-p}}.$$

Since $q - p \ge 1$, it follows that $|F| \le \frac{M}{R^k}$ where $M = \frac{|a_p|}{|b_q|}$ and $k > 0$. Theorem 7.15 then applies. □

Example 7.17 Evaluate $I = \displaystyle\int_0^\infty \frac{x\sin x}{x^2+a}\,dx$, where $a > 0$.

Since the integrand is even, $2I$ is the integral from $-\infty$ to ∞. Let $F(z) = \dfrac{z}{z^2+a}$, then $|F(z)| \sim \dfrac{1}{R}$ on Γ as $R \to \infty$, hence, Corollary 7.16 applies and the integral along Γ vanishes. Therefore, in the limit as $R \to \infty$,

$$2I = \operatorname{Im}\left\{\oint_C F(z)e^{iz}\,dz\right\} = \operatorname{Im}\left\{2\pi i\sum_k \operatorname*{Res}_{z=z_k} F(z)e^{iz}\right\} = 2\pi\operatorname{Re}\left\{\sum_k \operatorname*{Res}_{z=z_k} F(z)e^{iz}\right\},$$

where the z_k are the poles of $F(z)$ in the upper half-plane. The only such pole is $z = i\sqrt{a}$, and it is simple. Thus,

$$I = \pi\operatorname{Re}\left\{\operatorname*{Res}_{z=i\sqrt{a}} F(z)e^{iz}\right\}.$$

Now

$$\operatorname*{Res}_{z=i\sqrt{a}} F(z)e^{iz} = \lim_{z\to i\sqrt{a}}(z - i\sqrt{a})\frac{ze^{iz}}{(z-i\sqrt{a})(z+i\sqrt{a})} = \frac{i\sqrt{a}e^{-\sqrt{a}}}{i2\sqrt{a}} = \frac{e^{-\sqrt{a}}}{2}.$$

Therefore, $I = \dfrac{\pi}{2}e^{-\sqrt{a}}$.

Example 7.18 Evaluate $I = \displaystyle\int_{-\infty}^{\infty} \frac{\cos(4x)}{x^6+1}\,dx$.

Let $F(z) = \dfrac{1}{z^6+1}$. Then clearly Corollary 7.16 applies so that $\lim\limits_{R\to\infty}\displaystyle\int_{\Gamma} F(z)e^{i4z}\,dz = 0$. Therefore, we have

$$\begin{aligned} I &= \operatorname{Re}\left(\oint_C F(z)e^{i4z}\,dz\right) = \operatorname{Re}\left(2\pi i\sum_k \operatorname*{Res}_{z=z_k} F(z)e^{i4z}\right) \\ &= -2\pi \operatorname{Im}\left(\sum_k \operatorname*{Res}_{z=z_k} F(z)e^{i4z}\right), \end{aligned}$$

where z_k are the poles of $F(z)e^{i4z}$ in the upper half-plane. There are three such poles, namely

$$z_1 = e^{i\pi/6}, \qquad z_2 = e^{i\pi/2} = i, \qquad z_3 = e^{i5\pi/6};$$

all of which are simple. Computing the residue of the first pole we have

$$\begin{aligned} \operatorname*{Res}_{z=z_1} F(z)e^{i4z} &= \lim_{z\to z_1}(z-z_1)F(z)e^{i4z} = \lim_{z\to z_1}\frac{(z-z_1)e^{i4z}}{z^6+1} \\ &= \lim_{z\to z_1}\frac{e^{i4z}+(z-z_1)4ie^{4iz}}{6z^5} = \frac{e^{i4z_1}}{6z_1^5}. \end{aligned}$$

Similar calculations give $\operatorname*{Res}\limits_{z=z_2} F(z)e^{i4z} = \dfrac{e^{i4z_2}}{6z_2^5}$ and $\operatorname*{Res}\limits_{z=z_3} F(z)e^{i4z} = \dfrac{e^{i4z_3}}{6z_3^5}$. Summing these three residues is straight-forward but simplification takes some effort:

$$\begin{aligned} S &= \frac{\exp\left(i4(\sqrt{3}/2+i/2)\right)}{6e^{i5\pi/6}} + \frac{\exp\left(i4(i)\right)}{6e^{i5\pi/2}} + \frac{\exp\left(i4(-\sqrt{3}/2+i/2)\right)}{6e^{i25\pi/6}} \\ &= \frac{1}{6}\left[\exp\left(-2+i\left(2\sqrt{3}-\frac{5\pi}{6}\right)\right) - i\exp(-4) + \exp\left(-2+i\left(-2\sqrt{3}-\frac{\pi}{6}\right)\right)\right] \\ &= \frac{e^{-2}}{6}\left[\exp\left(i\left(2\sqrt{3}-\frac{5\pi}{6}\right)\right) - i\exp(-2) + \exp\left(-i\left(2\sqrt{3}+\frac{\pi}{6}\right)\right)\right]. \end{aligned}$$

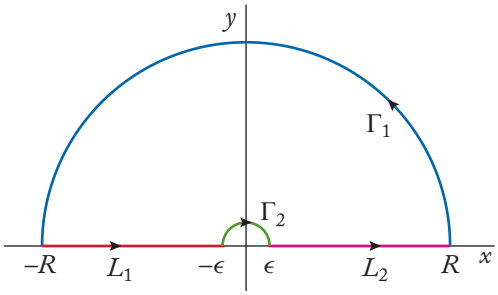


Figure 7.3: A contour with a vanishingly small semicircular arc to avoid a singularity at the origin.

The above can be simplified further by noting that $5\pi/6 = \pi - \pi/6$ and $e^{-i\pi} = -1$. Therefore,

$$S = \frac{e^{-2}}{6}\left[-\exp\left(i\left(2\sqrt{3}+\frac{\pi}{6}\right)\right) - i\exp(-2) + \exp\left(-i\left(2\sqrt{3}+\frac{\pi}{6}\right)\right)\right]$$
$$= \frac{e^{-2}}{6}\left[-2i\sin\left(2\sqrt{3}+\frac{\pi}{6}\right) - ie^{-2}\right].$$

Taking the imaginary part of S we have

$$\mathrm{Im}(S) = \frac{-e^{-2}}{6}\left[2\sin\left(2\sqrt{3}+\frac{\pi}{6}\right) - e^{-2}\right].$$

Finally,

$$I = -2\pi\,\mathrm{Im}(S) = \frac{\pi e^{-2}}{3}\left[2\sin\left(2\sqrt{3}+\frac{\pi}{6}\right) - e^{-2}\right].$$

Integrals with Improper Integrands

We can also do some improper real integrals that are improper due to both having unbounded limits of integration and having an integrand that is unbounded. For example, consider the integral

$$I = \int_0^\infty \frac{\sin(x)}{x}\,dx.$$

Since the integrand is even, $2I = \displaystyle\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx$.

To compute this integral we consider the path $C = \Gamma_1 + L_1 + \Gamma_2 + L_2$, where the contours Γ_j and L_j are shown in Fig. 7.3. The contour Γ_2 is a small half circle of radius ϵ that is introduced to avoid the singularity of the integrand at $z = 0$. By the Cauchy–Goursat theorem

we know that $\oint_C \frac{e^{iz}}{z}\,dz = 0$, and by Corollary 7.16 we know that $\lim_{R\to\infty} \int_{\Gamma_1} \frac{e^{iz}}{z}\,dz = 0$. The contours L_1 and L_2 will together approach the entire real axis as both $R \to \infty$ and $\epsilon \to 0$, so it only remains to consider the value of $\int_{\Gamma_2} \frac{e^{iz}}{z}\,dz$ in the limit as $\epsilon \to 0$.

To compute this last integral we may using the following result for vanishing arcs around simple poles.

Theorem 7.19 Vanishing Arcs Around Simple Poles. *If $f(z)$ has a simple pole at $z = z_0$ and Γ is the arc $\Gamma = z_0 + \epsilon e^{i\phi}$, $\phi_1 \le \phi \le \phi_2$, then*

$$\lim_{\epsilon\to 0} \int_\Gamma f(z)\,dz = i(\phi_2 - \phi_1) \operatorname*{Res}_{z=z_0} f(z).$$

Proof. The Laurent series for f valid on $0 < |z - z_0| < R$ for some R is

$$f(z) = \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n = \frac{a_{-1}}{z - z_0} + g(z),$$

where g is analytic. Therefore,

$$\int_\Gamma f(z)\,dz = \int_{\phi_1}^{\phi_2} \frac{a_{-1}}{\epsilon e^{i\phi}} \epsilon i e^{i\phi}\,d\phi + \int_\Gamma g(z)\,dz.$$

Now $\left|\int_\Gamma g(z)\,dz\right| \le \max_{z\in\Gamma} |g(z)|\,\epsilon(\phi_2 - \phi_1)$, which tends to 0 as ϵ vanishes. Therefore,

$$\lim_{\epsilon\to 0} \int_\Gamma f(z)\,dz = \int_{\phi_1}^{\phi_2} a_{-1} i\,d\phi + 0 = a_{-1} i(\phi_2 - \phi_1).$$

Since $\operatorname{Res}_{z=z_0} f(z) = a_{-1}$, this completes the proof. □

A note of caution that the above theorem is only valid for simple poles; there is no analogous theorem for poles with order greater than one.

Returning then to the example we have

$$\begin{aligned} 2I &= \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx = \operatorname{Im}\left(\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_{L_1+L_2} \frac{e^{iz}}{z}\,dz\right) \\ &= \operatorname{Im}\left(\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \int_C \frac{e^{iz}}{z}\,dz - \int_{\Gamma_1} \frac{e^{iz}}{z}\,dz - \int_{\Gamma_2} \frac{e^{iz}}{z}\,dz\right) \\ &= \operatorname{Im}\left(0 - 0 - i(0 - \pi) \operatorname*{Res}_{z=0} \frac{e^{iz}}{z}\right) = \operatorname{Im}(i\pi) = \pi. \end{aligned}$$

Therefore, $I = \pi/2$.

The trick of introducing vanishing circular arcs can also be applied to other singularities. Consider the integral

$$I = \int_0^\infty \frac{\ln x}{(x^2+1)^2}\,dx.$$

The integrand again has a singularity at $x = 0$ since the logarithm is undefined there. To compute this integral we may choose the same contour C as depicted in Fig. 7.3 and choose a branch of the complex logarithmic function such that the branch cut is outside C. In particular, we may take

$$\log(z) = \ln|z| + i\arg z, \qquad \text{where} \qquad -\frac{\pi}{2} < \arg z \le \frac{3\pi}{2}.$$

With this branch of the logarithm we define

$$f(z) = \frac{\log z}{(z^2+1)^2} = \frac{\log z}{(z-i)^2(z+i)^2}.$$

The only singularity of f inside C is $z = i$, which is a pole of order two. Therefore, by the Residue theorem,

$$\begin{aligned}\int_C f(z)\,dz &= 2\pi i \lim_{z\to i}\frac{d}{dz}\frac{\log z}{(z+i)^2} = 2\pi i \lim_{z\to i}\frac{\frac{1}{z}(z+i)^2 - 2(z+i)\log z}{(z+i)^4}\\ &= 2\pi i\left(\frac{4i - 4i\log(i)}{16}\right) = \frac{-\pi}{2}(1-\log(i)) = \frac{-\pi}{2}\left(1 - i\frac{\pi}{2}\right).\end{aligned}$$

The integral we are interested in is the integral of $f(z)$ along L_2 in the limit as $R\to\infty$ and $\epsilon\to 0$. We therefore need to know the integrals along the other arcs that make up C. Along Γ_1 the integrand satisfies

$$\lim_{R\to\infty}\left|\frac{\ln R + i\theta}{(R^2e^{i2\theta}+1)^2}\right| = \lim_{R\to\infty}\frac{\left|\frac{1}{R}\right|}{\left|4Re^{i2\theta}(R^2e^{i2\theta}+1)\right|} \approx \frac{1}{4R^4}.$$

Therefore, by Theorem 7.11 the integral of f along Γ_1 is zero in the limit as $R\to\infty$. On Γ_2 we have

$$\int_{\Gamma_2} f(z)\,dz = \int_\pi^0 \frac{\ln\epsilon + i\theta}{(\epsilon^2e^{i2\theta}+1)^2}\epsilon i e^{i\theta}\,d\theta,$$

but

$$\lim_{\epsilon\to 0}\epsilon\ln\epsilon = \lim_{\epsilon\to 0}\frac{\ln\epsilon}{1/\epsilon} = \lim_{\epsilon\to 0}\frac{1/\epsilon}{-1/\epsilon^2} = 0.$$

We conclude that $\lim_{\epsilon\to 0}\int_{\Gamma_2} f(z)\,dz = 0$. For the portion along $L_1 = xe^{i\pi}$, $\epsilon < x < R$, we have

$$\begin{aligned}\int_{L_1} f(z)\,dz &= \int_R^\epsilon \frac{\log(xe^{i\pi})}{(x^2e^{i2\pi}+1)^2}e^{i\pi}\,dx = \int_\epsilon^R \frac{\log(x)+i\pi}{(x^2+1)^2}\,dx \\ &= \int_\epsilon^R \frac{\log x}{(x^2+1)^2} + i\pi\int_\epsilon^R \frac{1}{(x^2+1)^2}\,dx \to I + i\pi\int_0^\infty \frac{1}{(x^2+1)^2}\,dx.\end{aligned}$$

This last integral may be computed using the methods described above. The reader should verify that

$$\int_0^\infty \frac{1}{(x^2+1)^2}\,dx = \frac{\pi}{4}.$$

Given these results we may now compute I from

$$\begin{aligned}I &= \int_{L_2} f(z)\,dz = \oint_C f(z)\,dz - \int_{\Gamma_1} f(z)\,dz - \int_{\Gamma_2} f(z)\,dz - \int_{L_1} f(z)\,dz \\ &= \frac{-\pi}{2}\left(1 - i\frac{\pi}{2}\right) - 0 - 0 - \left(I + i\pi\frac{\pi}{4}\right) = -I - \frac{\pi}{2}.\end{aligned}$$

Therefore, $I = -\dfrac{\pi}{4}$.

7.3.3 EXERCISES

7.7. Compute the following real integrals using the Residue theorem:

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$, **(b)** $\displaystyle\int_0^{2\pi} \frac{d\theta}{4-2\sin\theta-\cos\theta}$,

(c) $\displaystyle\int_0^{2\pi} \frac{\sin\theta}{(5-2\sin\theta)^2}\,d\theta$, **(d)** $\displaystyle\int_0^{2\pi} \frac{\cos^2\theta}{3+\sin\theta}\,d\theta$.

7.8. Use the residue theorem to compute the following real integrals, where $a > 0$.

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+16)(x^2+3)}\,dx$, **(b)** $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+a^2)^3}\,dx$, **(c)** $\displaystyle\int_0^{\infty} \frac{x^3\sin x}{ax^4+1}\,dx$.

7.9. Show that

$$\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\,dx = \sec(1),$$

by integrating $e^{2z}/\cosh(\pi z)$ around rectangles with vertices at $z = \pm R$, $R+i$, and $-R+i$, where $R > 0$.

7.10. By using a contour that has small indentations around the singularities on the real line, compute

$$I = \int_{-\infty}^{\infty} \frac{x\cos(x)}{x^2 - 3x + 2}\, dx.$$

7.11. Show that

$$\int_0^{\infty} \frac{x^{\alpha}}{x^2 - 1}\, dx = \frac{\pi\,[1 - \cos(\alpha\pi)]}{2\sin(\alpha\pi)},$$

where $-1 < \alpha < 1$, and $\alpha \neq 0$. [Hint: Set $z^{\alpha} = e^{\alpha \log z}$ and choose the branch cut of log to lie on the positive real axis. Then consider a contour composed of a large circle centered at the origin, a small circle centered at the origin and two contours along the top and bottom of the branch cut, both with an indentation around $z = 1$.]

CHAPTER 8

Conformal Mapping

In this chapter we give a short introduction to conformal mapping. More extensive coverage of this topic can be found in Wunsch [2005] and Churchill and Brown [1990].

8.1 CONFORMAL MAPS

Look back at the figures in Chapter 2 where regions R in $\mathbb{C}$ were mapped to their images under various analytic functions $f(z)$. In each case notice that the curves in R that crossed at right angles also crossed at right angles in the image. As we shall see in this section, this property is generally true for curves that cross at any angle.

Let Γ_1 be a smooth arc with parameterization $z(t)$, $t \in [a, b]$, and let f be an analytic function on Γ_1. The image, C_1, of Γ_1 under the transformation $w = f(z)$ is given parametrically by $w(t) = f(z(t))$, $t \in [a, b]$. By the chain rule we have

$$w'(t) = f'(z(t))\, z'(t).$$

Now since $z(t)$ is smooth, by definition $z'(t) \neq 0$ for $t \in [a, b]$. For some $t_0 \in [a, b]$ assume that $z_0 = z(t_0)$ is a point such that $f'(z_0) \neq 0$. Then both factors on the right-hand side of the above equation have a well-defined argument. Therefore,

$$\arg(w'(t_0)) = \arg(f'(z_0)) + \arg(z'(t_0)). \tag{8.1}$$

Now $z'(t_0)$ is the tangent vector to Γ_1 at z_0 and $w'(t_0)$ is the tangent vector to C_1 at $w_0 = f(z_0)$ (see Section 4.1.1), thus $\arg(z'(t_0))$ and $\arg(w'(t_0))$ are the directions of Γ_1 and C_1 at z_0 and w_0, respectively. The above equation tells us that the direction of C_1 at w_0 is rotated by $\arg(f'(z_0))$ compared to the direction of Γ_1 at z_0. But this same argument also holds for another smooth arc Γ_2 and its image C_2. Therefore, if Γ_1 and Γ_2 intersect at a point z_0 with angle ϕ, then their images C_1 and C_2 intersect at w_0 with the same angle ϕ, since both C_1 and C_2 have been rotated by the same amount, namely $\arg(f'(z_0))$. Mappings with this property are called conformal.

Definition 8.1 Conformal Map. A mapping $w = f(z)$ that preserves both the magnitude and sense of the angle of intersection between any two smooth curves intersecting at z_0 is said to be a **conformal** mapping at z_0. If f is conformal at every point in a domain D then f is said to be conformal in D.

Definition 8.2 Isogonal Map. A mapping that preserves the magnitude but not necessarily the sense of the angle of intersection is called an **isogonal** mapping.

Our discussion above is a proof of the following theorem.

Theorem 8.3 *If f is analytic in a domain D then f is conformal at each point z in D where $f'(z) \neq 0$.*

Example 8.4 Show explicitly that the function $f(z) = 1/(z+1)$ preserves the magnitude and sense of the angle of intersection of the arcs Γ_1 and Γ_2 with parameterizations $z_1(t) = t + it$ and $z_2(t) = 1 + it$.

Clearly, Γ_1 is the line $x = y$ and Γ_2 is the vertical line $x = 1$, which intersect at the point $z = 1 + i$ with angle $\pi/4$ (measuring from positive Γ_1 direction to positive Γ_2 direction). This intersection occurs when $t = 1$ in both parameterizations. The images of Γ_1 and Γ_2 under f are

$$C_1: \quad w_1(t) = \frac{1}{2+it}, \qquad C_2: \quad w_2(t) = \frac{1}{1+t+it}, \qquad t \in [a,b].$$

The tangent vectors to these at $t = 1$ are

$$w_1'(t)\big|_{t=1} = \left.\frac{-i}{(2+it)^2}\right|_{t=1} = \frac{-i}{3+4i} = \frac{-4-3i}{25},$$

$$w_2'(t)\big|_{t=1} = \left.\frac{-(1+i)}{(1+t+it)^2}\right|_{t=1} = \frac{-(1+i)}{3+4i} = \frac{-7+i}{25}.$$

The ratio of these two vectors is

$$\frac{w_2'(1)}{w_1'(1)} = \frac{-4-3i}{-7+i} = \frac{(-4-3i)(-7-i)}{50} = \frac{25+25i}{50} = \frac{1+i}{2} = \frac{1}{\sqrt{2}}\,e^{i\pi/4}.$$

Thus, when $w_1'(1)$ is multiplied by $\frac{1}{\sqrt{2}}\,e^{i\pi/4}$ it becomes $w_2'(1)$. Recall that multiplication scales and rotates, so the angle from $w_1'(1)$ to $w_2'(1)$ is $\pi/4$.

Example 8.5 Show that the map $f(z) = \overline{z}$ is isogonal but not conformal.

Since the action of conjugation is simply reflection across the real axis, it is clear that the magnitude of all angles between two intersecting curves is preserved. However, the sense is reversed, since if Γ_1 has direction θ_1 and Γ_2 has direction θ_2 at a point of intersection z_0, then the images of Γ_1 and Γ_2 will have directions $-\theta_1$ and $-\theta_2$ at $f(z_0)$. So the angle from Γ_1 to Γ_2 is $\phi = \theta_2 - \theta_1$, but the angle from the image of Γ_1 to the image of Γ_2 is $\psi = (-\theta_2) - (-\theta_1) = -\phi$.

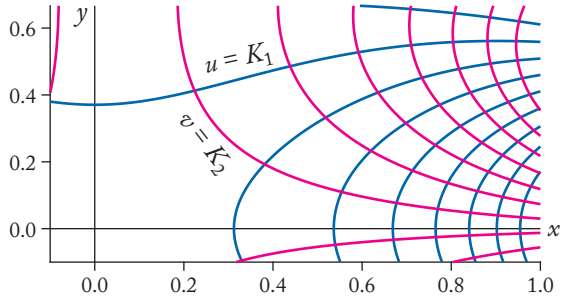


Figure 8.1: Level sets for the real and imaginary parts of $f(z) = e^{z^2}$.

That analytic functions are conformal allows us to easily show a special geometric feature of harmonic conjugates. Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function in a domain D, then v is the harmonic conjugate of u. For any constants K_1 and K_2, the level sets $u(x, y) = K_1$ and $v(x, y) = K_2$ in D are mapped by $w = f(z)$ to vertical and horizontal lines in the w-plane, respectively. Since horizontal and vertical lines are orthogonal to each other at points of intersection, it follows from the conformal nature of f that their preimages $u(x, y) = K_1$ and $v(x, y) = K_2$ are orthogonal at points of intersection. Therefore, the level sets of u are orthogonal to the level sets of v.

Example 8.6 Consider the analytic function $f(z) = e^{z^2} = u(x, y) + iv(x, y) = e^{x^2-y^2}\cos(2xy) + ie^{x^2-y^2}\sin(2xy)$. Level sets $u = K_1$ and $v = K_2$ for various constants K_1 and K_2 are shown in Fig. 8.1. Clearly, the level sets $u = K_1$ cross the level sets $v = K_2$ orthogonally.

8.1.1 EXERCISES

8.1. Show that the function $f(z) = \sin z$ is conformal at all points except $(2n + 1)\pi/2$, $n \in \mathbb{Z}$. Sketch the image under f of the semi-infinite strip $-\pi/2 \le x \le \pi/2$, $y \ge 0$ and note how the preservation of angle is lost at the points where the mapping is not conformal.

8.2. Show that the map

$$w = f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

maps the portion of the upper half-plane outside the unit circle to the upper half-plane. That is, it maps $R = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0 \text{ and } |z| > 1\}$ to $R' = \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$. Where is this map not conformal?

8.3. Show that the map

$$w = f(z) = \frac{1}{z}$$

maps the disk radius R centered at $z = R$ to the half-plane $w > 1/(2R)$.

8.2 APPLICATION TO LAPLACE'S EQUATION

One of the applications of conformal mapping is to the solution of problems involving Laplace's equation on an open connected two-dimensional region, Ω, with some prescribed conditions on the boundary, $\partial\Omega$. The problem

$$\begin{aligned} h_{xx} + h_{yy} &= 0 \qquad && (x, y) \in \Omega \subset \mathbb{R}^2, \\ h(x, y) &= h_0(x, y) && (x, y) \in \partial\Omega \end{aligned} \tag{8.2}$$

is called a Dirichlet problem, which is Laplace's equation for the function h holding on Ω and the value of h specified on the boundary $\partial\Omega$.[1] The boundary condition itself is called a Dirichlet boundary condition. If the boundary condition is replaced with a specification of the derivative of h in the direction normal to the boundary,

$$\frac{\partial h}{\partial n}(x, y) = h_0(x, y) \qquad (x, y) \in \partial\Omega, \tag{8.3}$$

the problem is called a Neumann problem and the boundary condition is called a Neumann condition.[2] There is a third type of problem with a mixed, or Robin, boundary condition:

$$\alpha h(x, y) + \beta \frac{\partial h}{\partial n}(x, y) = h_0(x, y) \qquad (x, y) \in \partial\Omega, \tag{8.4}$$

where α and β are real constants.[3]

The boundary function $h_0(x, y)$ can be quite general, but we shall only deal with the special cases of a Dirichlet condition with h_0 being a piecewise constant function, or a problem where a constant Dirichlet condition is imposed on one portion of the boundary and a zero Neumann condition on another. This latter type of condition could be considered a Robin condition where α and β are generalized to be piecewise constants with one being 0 and the other 1, at each point on the boundary.

Sometimes it is more convenient to express the problem in terms of polar coordinates. In this case, Laplace's equation reads

$$h_{rr} + \frac{1}{r} h_r + \frac{1}{r^2} h_{\theta\theta} = 0. \tag{8.5}$$

[1] Peter Gustav Dirichlet (1805–1859) was a German mathematician who was the founder of analytic number theory and made contributions in the theory of Fourier series and mathematical analysis.

[2] Carl Neumann (1832–1925) was a German mathematician who worked on electrodynamic theory.

[3] Victor Robin (1855–1897) was a French applied mathematician working primarily in the area of mathematical physics.

Problems involving Laplace's equation arise in a number of application areas including steady-state temperature profiles, electrostatics, and velocity potentials in fluid flow. We shall concentrate on the temperature situation. If h is the temperature of some essentially two-dimensional material (for example, a thin plate with insulated upper and lower faces) that occupies a region Ω in the plane then the governing heat equation is

$$h_t = h_{xx} + h_{yy}, \qquad (x, y) \in \Omega, \quad t > 0.$$

(For a derivation of this equation, see, for example, Boyce and DiPrima [2001, p. 614].) If a steady-state solution is sought, one that does not change in time, then the time derivative is zero and the equation collapses to Laplace's equation on Ω. It can be shown that if the temperature or its normal derivative is specified on the boundary then there is a unique solution to this problem.

In general, obtaining closed form solutions to these problems, even with our restriction on h_0 being zero or a constant, can be very difficult or impossible, depending on the geometry of Ω. Conformal maps can play a role in such problems as follows. We identify the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ through $z = x + iy$ and thus Ω can refer to either a region of $\mathbb{R}^2$ or the corresponding region of $\mathbb{C}$. Since solutions of Laplace's equation in two dimensions are harmonic functions, they can be thought of as the real part of an analytic function $f(z)$. If a conformal map $w = g(z)$ can be found that maps Ω to a region R on which the boundary conditions are in a simpler arrangement, then it may be possible to solve the problem in terms of $w = u + iv$ on R and then interpret the solution back in terms of x and y.

First, we will show that conformal mapping takes harmonic functions to harmonic functions, which follows from the fact that harmonic conjugates exist, and the composition of analytic functions is analytic.

Theorem 8.7 *If $H(u, v)$ is a harmonic function on a simply connected domain R and if $f(z) = u(x, y) + iv(x, y)$ is an analytic function that maps Ω to R, then $h(x, y) = H(u(x, y), v(x, y))$ is a harmonic function on Ω.*

Proof. Since R is simply connected, by Theorem 5.24, there exists a function G that is the harmonic conjugate of H on R. Let $g(w) = H(u, v) + iG(u, v)$, where $w = u + iv$, then g is an analytic function on R. Since f is also analytic and maps Ω to R, it follows that the composition $(g \circ f)(z) = g(f(z))$ is analytic on Ω. (This last claim follows from the chain rule for differentiation.) Therefore, the real part of the composition,

$$\operatorname{Re}[g(f(z))] = \operatorname{Re}[g(u(x, y) + iv(x, y))] = H(u(x, y), v(x, y)),$$

is harmonic on Ω. □

The theorem also holds without the requirement that R be simply connected, but the proof is more difficult. See, for example, Churchill and Brown [1990, pp. 249–250].

Next, we consider how boundary conditions transform under a conformal map.

Theorem 8.8 *If $f(z) = u(x, y) + iv(x, y)$ is an analytic function that conformally maps a smooth arc Γ to an arc C, and if H is a function defined on C that satisfies either of the conditions*

$$H(u, v) = h_0, \qquad \text{or} \qquad \frac{\partial H}{\partial N}(u, v) = 0, \qquad \forall (u, v) \text{ on } C,$$

where h_0 is a real constant and N is a direction normal to C, then the function $h(x, y) = H(u(x, y), v(x, y))$ satisfies the corresponding conditions

$$h(x, y) = h_0, \qquad \text{or} \qquad \frac{\partial h}{\partial n}(x, y) = 0, \qquad \forall (x, y) \text{ on } \Gamma,$$

where n is a direction normal to Γ.

Proof. By definition of h, the level set $h(x, y) = h_1$, where h_1 is some real constant, is mapped by f to the corresponding level set of H, namely $H(u, v) = h_1$. Conversely, if (u, v) is a point such that $H(u, v) = h_1$, and if $w = u + iv$ is in the range of f, then there exists a $z = x + iy$ such that $f(z) = w$ and $h(x, y) = h_1$.

Suppose $H = h_0$ for all (u, v) on C. Thus, C is a level set of H. Since f maps Γ to C, it immediately follows that $h = h_0$ for all (x, y) on Γ.

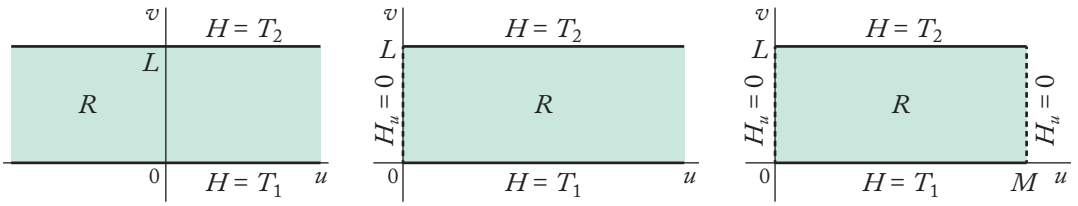
Suppose $\frac{\partial H}{\partial N} = 0$ for all (u, v) on C. From calculus we know that the normal derivative is equal to $\nabla H \cdot N$, where $\nabla H = \begin{bmatrix} H_u \\ H_v \end{bmatrix}$ and N is a unit vector normal to C. Further, we know that ∇H is orthogonal to level sets of H and if the dot product of two vectors is zero, they are orthogonal. Since $\frac{\partial H}{\partial N}$ is zero, it follows that either $\nabla H = 0$ or ∇H is orthogonal to N. In the former case, by the definition of h and direct computation it follows that h_x and h_y are zero, hence, $\frac{\partial h}{\partial n} = 0$. In the latter case, since ∇H is orthogonal to N, level sets of H are parallel to N and therefore orthogonal to C. Since f is a conformal mapping on Γ (and therefore also in a open set containing Γ) it must preserve angles. Since Γ maps to C it follows that the level sets of h (which we know map to the level sets of H) must be orthogonal to Γ, which means $\frac{\partial h}{\partial n} = 0$ on Γ. □

Although Neumann conditions with zero normal derivative transform to the same under an analytic map, it is not true when $\frac{\partial h}{\partial n}$ is a nonzero constant, hence, our restriction to the normal derivative being zero.

We now examine several different geometries R in the (u, v)-plane and arrangements of boundary conditions of the type in Theorem 8.8 that yield simple solutions for a harmonic function $H(u, v)$ on R satisfying these conditions. In each case the problem is solved by a solution

of Laplace's equation that is independent of one of the two variables. For Laplace's equation in Cartesian coordinates, $H_{uu} + H_{vv} = 0$, a solution independent of u is clearly $H(u, v) = a + bv$, where a and b are real constants. Similarly, $H(u, v) = a + bu$ is also a solution. In the case of polar coordinates $w = \rho e^{i\phi}$, Laplace's equation reads $H_{\rho\rho} + \frac{1}{\rho} H_\rho + \frac{1}{\rho^2} H_{\phi\phi} = 0$. A solution independent of ρ is $H(\rho, \phi) = a + b\phi$. If H does not depend on ϕ, the general solution of Laplace's equation is $H(\rho, \phi) = a + b \ln \rho$. In the figures below, a thick solid line represents a Dirichlet condition and a thick dotted line represents a zero Neumann condition. If there is no line bordering the region, the region extends to infinity.

Consider the cases of an infinite strip, a semi-infinite strip, and a rectangle as shown in the diagram below.

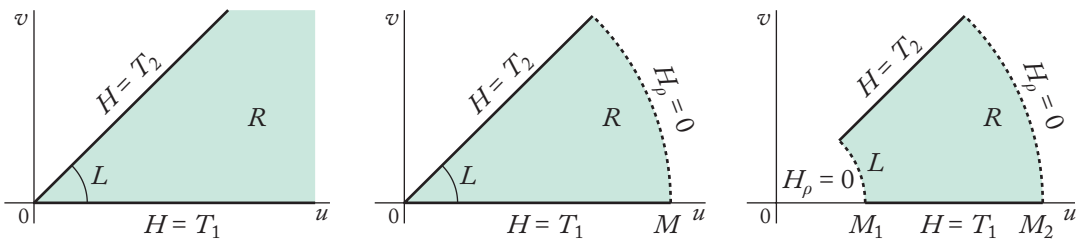


In each of these cases the solution is

$$H(u, v) = T_1 + (T_2 - T_1)\frac{v}{L}. \tag{8.6}$$

The zero Neumann conditions along the vertical boundaries is what allows this solution to hold in the semi-infinite strip and rectangle cases. Of course the strips could be reflected in the line $u = v$ and the solution given in terms of u rather than v.

Now consider the cases where the region is a sector, infinite or finite, with the boundary conditions as specified in the diagram below.



In each of these cases the solution is

$$H(\rho, \phi) = T_1 + (T_2 - T_1)\frac{\phi}{L}, \tag{8.7}$$

which, in Cartesian coordinates, is

$$H(u,v) = T_1 + \frac{T_2 - T_1}{L}\,\mathrm{atan2}_{[0,2\pi)}(v,u). \tag{8.8}$$

If $L = \pi$ in the first of these three cases, then R is the entire upper half-plane, in which case it may be more natural to express the solution in terms of Cartesian coordinates (8.8). In this case, since the range of angles for the upper half-plane is $[0,\pi]$, and in view of the result of Exercise 1.4, we may shift the interval for atan2 to the standard one and write

$$H(u,v) = T_1 + \frac{T_2 - T_1}{\pi}\,\mathrm{atan2}(v,u). \tag{8.9}$$

One could also consider cases where the region R is an annulus or partial annulus with constant temperatures on the circular arcs. However, since the mapping $w = \log(z)$ maps such regions to a rectangle, which we have already included, it is not necessary to consider these separately.

In the following examples, we illustrate in a few simple cases how a conformal map can be used to transform a given region Ω to one of the regions R above, allowing for a solution of Laplace's equation and the specified boundary conditions on Ω. We make use of linear fractional transformations, which map circles to circles, and in particular the formula given by Eq. (2.7), which determines a linear fractional transformation that maps a given set of three points (defining one circle) to another set of three points (defining a second circle).

Example 8.9 Consider the steady-state heat problem on a disk Ω of radius 1 where the temperature is held at constant 3 along one half of the boundary and a temperature 0 along the other half:

$$h_{rr} + \frac{1}{r}h_r + \frac{1}{r^2}h_{\theta\theta} = 0, \qquad r < 1,$$
$$h(1,\theta) = \begin{cases} 3 & \text{if } 0 \le \theta \le \pi, \\ 0 & \text{if } \pi < \theta < 2\pi. \end{cases}$$

This problem is identical to the one in Example 5.31 where the solution was obtained by using Poisson's integral formula of the circle. Here we solve the problem by conformal mapping. Using Eq. (2.7), the linear fractional transformation that maps the three points on the boundary -1, i, and 1 to the three points ∞, 1, and 0 is $w = f(z) = -i\frac{z-1}{z+1}$. We can see that the interior of the disk Ω maps to the upper half-plane R by noting that 0 maps to i. Points on the lower half of the circle map to the negative real axis and points on the upper half of the circle map to the positive real axis. See Fig. 8.2. Thus, the problem transforms to one on R where R is a sector with angle π. Using (8.9) the solution on R is

$$H(u,v) = 3 + \frac{-3}{\pi}\,\mathrm{atan2}(v,u), \qquad v > 0, \qquad \text{where } f = u + iv.$$

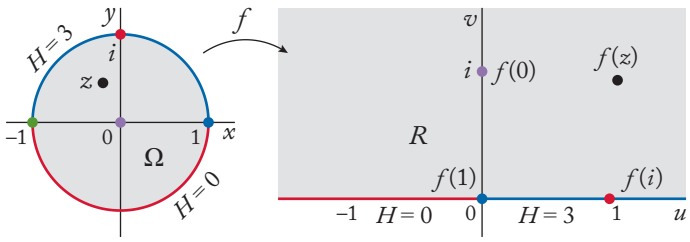


Figure 8.2: The region Ω (left) and its image R (right) under mapping f from Example 8.9.

To compute u and v in terms of r and θ we may write

$$f = -i\frac{z-1}{z+1} = -i\frac{(z-1)(\overline{z}+1)}{(z+1)(\overline{z}+1)} = -i\frac{z\overline{z}+z-\overline{z}-1}{z\overline{z}+z+\overline{z}+1}$$
$$= -i\frac{x^2+y^2-1+i2y}{(x+1)^2+y^2} = \frac{2y+i(1-x^2-y^2)}{(x+1)^2+y^2}.$$

Therefore,

$$u = \frac{2y}{(x+1)^2+y^2}, \qquad v = \frac{1-x^2-y^2}{(x+1)^2+y^2}.$$

Recall that $\text{atan2}(ca, cb) = \text{atan2}(a, b)$ for any positive real constant c. Therefore, $\text{atan2}(v, u) = \text{atan2}(1-x^2-y^2, 2y)$. Finally, since $x^2+y^2 = r^2$ and $y = r\sin\theta$,

$$h(r,\theta) = 3 - \frac{3}{\pi}\,\text{atan2}(1-r^2, 2r\sin\theta).$$

This solution should be compared with that in Example 5.31. Although the solutions look substantially different, they are in fact the same function.

In the next example, a logarithm and a linear fractional transformation are used to map Ω to R.

Example 8.10 Consider steady-state temperature in a semi-infinite plate where one edge is held at a constant temperature T_1 except for a gap of width two in which the plate is held

constant at T_2. The problem is thus

$$h_{xx} + h_{yy} = 0, \qquad y > 0,$$
$$h(x,0) = \begin{cases} T_2 & \text{if } |x| < 1, \\ T_1 & \text{otherwise.} \end{cases}$$

We claim that the mapping $w = f(z) = \log\left(\frac{z-1}{z+1}\right)$ maps the upper half-plane, Ω, to the infinite strip $R = \{w = u + iv \in \mathbb{C} \,:\, 0 < v < \pi\}$ as illustrated in Fig. 8.3. Further, the interval $(-1, 1)$ on the real axis is mapped to the line $v = \pi$ and the rest of the real axis is mapped to the line $v = 0$. This can be seen as follows. First, the linear fractional transformation $\zeta = (z-1)/(z+1)$ maps the points -1, 0, and 1 to ∞, -1, and 0, respectively. Both of these three-point sets define a line. Thus, we immediately see the real axis maps to the real axis, and that the interval $(-1, 1)$ maps to the negative real axis, while the points on the real axis outside the interval $(-1, 1)$ map to the positive real axis. (This can also be seen by letting $z = x$ in the expression for ζ.) Then applying the logarithm to the upper half ζ-plane will send it to the infinite strip R since the arguments of points in the upper half-plane are between 0 and π, and the logarithm of their moduli span all real numbers. The positive real axis in the ζ-plane maps to the line $v = 0$ while the negative real axis maps to the line $v = \pi$. Now, we should be careful which branch of the logarithm we choose. Since we require an analytic function on the upper half-plane, we should select a branch where the branch cut is out of the way, for example:

$$\log \zeta = \ln|\zeta| + i \arg(\zeta), \qquad -\frac{\pi}{2} < \arg(\zeta) \le \frac{3\pi}{2}.$$

With this choice, f is analytic on the entire upper half-plane, and on its boundary except for the points 1 and -1. Therefore, on the strip R we have $H = T_1$ on $v = 0$, and $H = T_2$ on $v = \pi$. Using (8.6), the solution to the transformed Dirichlet problem on R is

$$H(u,v) = T_1 + \frac{T_2 - T_1}{\pi} v.$$

But, using the same scaling property of atan2 as in Example 8.9, we have

$$v = \operatorname{Im}(f) = \operatorname{Im}\left(\log\left(\frac{z-1}{z+1}\right)\right) = \arg\left(\frac{z-1}{z+1}\right) = \arg\left(\frac{x^2+y^2-1+i2y}{(x+1)^2+y^2}\right)$$
$$= \operatorname{atan2}(2y, x^2+y^2-1).$$

Therefore,

$$h(x,y) = T_1 + \frac{T_2 - T_1}{\pi} \operatorname{atan2}(2y, x^2+y^2-1).$$

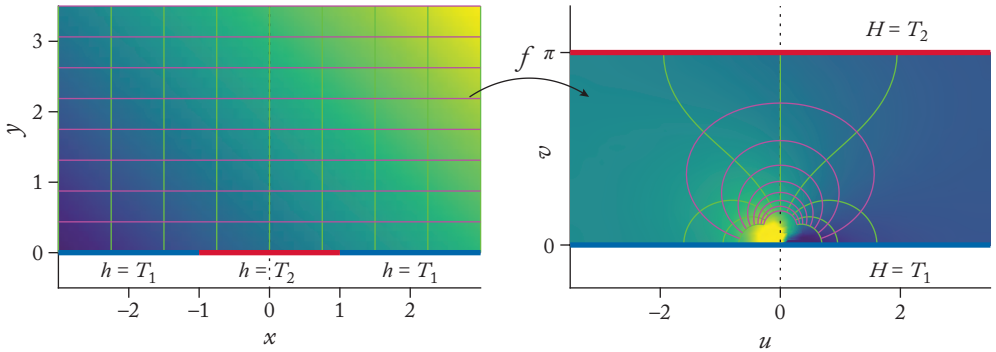


Figure 8.3: The region Ω (left) and its image R (right) under the mapping $f(z) = \log\left(\frac{z-1}{z+1}\right)$ from Example 8.10.

In our last example, a zero Neumann condition is imposed on one part of the boundary.

Example 8.11 Consider the steady-state heat problem on the infinite quadrant $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0,\ y > 0\}$ given by

$$\begin{aligned} h_{xx} + h_{yy} &= 0, & (x, y) &\in \Omega, \\ h(0, y) &= T_1, & y &> 0, \\ h(x, 0) &= T_2, & x &> 1, \\ h_y(x, 0) &= 0, & 0 &< x < 1. \end{aligned}$$

The function $w = f(z) = \arcsin z$ maps Ω to the semi-infinite strip $R = \{z \in \mathbb{C} : 0 < x < 1,\ y > 0\}$, such that the boundary $x = 0,\ y > 0$ of Ω is mapped to the positive imaginary axis, the boundary $x > 1,\ y = 0$ of Ω is mapped to the vertical line segment $u = \pi/2,\ v > 0$, and the boundary $0 < x < 1,\ y = 0$ of Ω is mapped to the horizontal line segment $0 < u < \pi/2,\ v = 0$. This is illustrated in Fig. 8.4. The transformed problem can then be solved using (8.6) (with u replacing v) yielding

$$H(u, v) = T_1 + \frac{T_2 - T_1}{\pi/2} u.$$

Our task is now to extract the real part u of $w = f(z) = \arcsin z$. Starting with $z = \sin w$ where $w = u + iv$ we find

$$x = \sin u \cosh v, \qquad y = \cos u \sinh v.$$

From this it follows that

$$\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1,$$

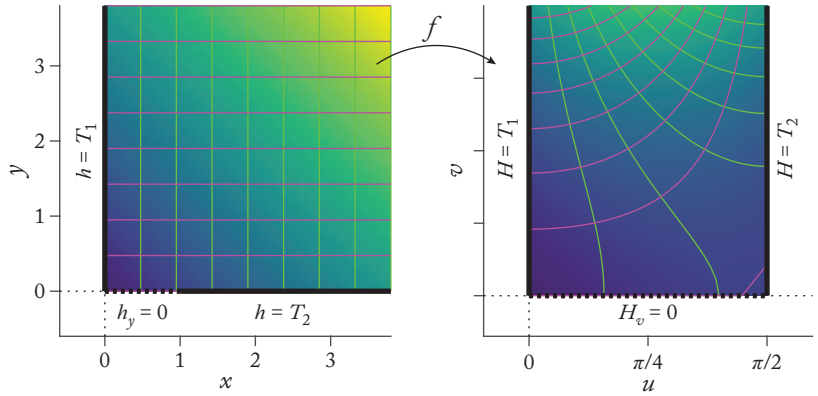


Figure 8.4: The region Ω (left) and its image R (right) under the mapping $f(z) = \arcsin z$ from Example 8.11.

which, for each fixed u, with $0 < u < \pi/2$, is a hyperbola with foci at $c = \pm\sqrt{\sin^2 u + \cos^2 u} = \pm 1$ on the x-axis. For a hyperbola, the absolute difference of the distances between any point on the hyperbola and the two foci is a constant equal to twice the semi-major axis, $\sin u$. Therefore,

$$\left|\sqrt{(x-1)^2 + y^2} - \sqrt{(x+1)^2 + y^2}\right| = 2\sin u,$$

and, since we require $x > 0$, this gives

$$u = \arcsin\left(\frac{1}{2}\left(\sqrt{(x-1)^2 + y^2} - \sqrt{(x+1)^2 + y^2}\right)\right).$$

Therefore, the solution to the original problem is

$$h(x, y) = T_1 + \frac{T_2 - T_1}{\pi}\arcsin\left(\frac{1}{2}\left(\sqrt{(x-1)^2 + y^2} - \sqrt{(x+1)^2 + y^2}\right)\right).$$

8.2.1 EXERCISES

8.4. Find a harmonic function, h, on the half disk of radius 1, centered at the origin, and in the upper half-plane, such that $h = T_1$ on the semi-circle part of the boundary, and $h = T_2$ on the x-axis part of the boundary. [Hint: Consider what the transformation in Example 8.9 does to the half disk.]

8.5. Find a harmonic function, h, on the half disk of radius 1, centered at the origin, and in the upper half-plane, such that $h = T_1$ on the portion of the boundary in the left half-plane, and $h = T_2$ on the portion of the boundary in the right half-plane. [Hint: Consider the function $f(z) = z + 1/z$.]

8.6. Find a harmonic function, h, on the semi-infinite strip $\Omega = \{(x, y) : |x| < \pi/2,\ y > 0\}$ such that $h = T_1$ on $x = \pm\pi/2,\ y > 0$, and $h = T_2$ on $|x| \leq \pi/2,\ y = 0$. [Hint: See problem 8.1 and use the solution from Example 8.10.]

8.7. Consider a region Ω bounded on the inside by a circle of radius R_1 centered at $(R_1, 0)$ and on the outside by a circle of radius $R_2 > R_1$ centered at $(R_2, 0)$. Suppose that the inner circle boundary is held at temperature T_1 and the outer circle boundary is held at temperature T_2. Find a harmonic function on Ω satisfying these boundary conditions. [Hint: Use a linear fractional transformation that maps 0 to ∞ (so that both circles will become lines) and maps $z_1 = 2R_1$ and $z_2 = 2R_2$ to $w_1 = 0$ and $w_2 = 1$.]

AND BEYOND...

Students wishing to pursue studies on complex analysis at a higher level are referred to the text "Functions of a Complex Variable," Carrier et al. [1983].

APPENDIX A

Greek Alphabet

Symbol	Name	Some common mathematical uses
A, α	alpha	α: alternate for a; real part of an eigenvalue
B, β	beta	β: alternate for b; imaginary part of an eigenvalue
Γ, γ	gamma	Γ: a curve; the Gamma function. γ: the Lorentz factor
Δ, δ	delta	Δ: a difference; the Laplacian operator. δ: a small positive number; the Kronecker delta function
E, ϵ	epsilon	ϵ: a small positive number
Z, ζ	zeta	ζ: alternate for z; a complex number; the Riemann zeta function
H, η	eta	η: alternate for y; viscosity
Θ, θ	theta	Θ: an angle; an asymptotically tight bound. θ: an angle; a temperature
I, ι	iota	
K, κ	kappa	κ: alternate for k; the condition number of a matrix
Λ, λ	lambda	Λ: a diagonal matrix of eigenvalues. λ: an eigenvalue
M, μ	mu	μ: an eigenvalue; a parameter; a mean
N, ν	nu	ν: alternate for n; a parameter
Ξ, ξ	xi	ξ: alternate for x
O, o	omicron	
Π, π	pi	Π: the product operator; a projection. π: the ratio of the circumference to the diameter of a circle; a projection
P, ρ	rho	ρ: alternate for r; density
Σ, σ	sigma	Σ: the summation operator; a diagonal matrix of singular values; the covariance matrix. σ: alternate for s; a permutation; a standard deviation
T, τ	tau	τ: alternate for t; a nondimensional time
Υ, υ	upsilon	
Φ, ϕ	phi	Φ: an angle. ϕ: an angle; a function; the golden ratio
X, χ	chi	χ: a statistical distribution; the graph chromatic number
Ψ, ψ	psi	Ψ: an angle. ψ: an angle; the wave function in quantum mechanics
Ω, ω	omega	Ω: a domain; an asymptotic lower bound. ω: an angular frequency

APPENDIX B

Answers to Selected Exercises

1.1 (e) all points on and right of the line $x = 1$; (f) the second and fourth quadrants. **1.2** (a) $2 - 2i$; (b) $-5 - i$; (c) $-26 - 7i$. **1.3** $z^2 = 2\operatorname{cis}\pi/2$, $z^3 = s\sqrt{2}\operatorname{cis}3\pi/4$. **1.5** the circle radius e^{x_0}, the ray with angle y_0. **1.7** $3\sqrt{3}e^{i\pi/2}$. **1.8** $n = 1 + 6k$, $k \in \mathbb{Z}$. **1.9** (a) $-\sqrt{2} - i\sqrt{2}$; (b) $2\sqrt{2}e^{i3\pi/4}$. **1.10** (a) $64\sqrt{3} + 64i$; (b) $-10 - i6\sqrt{3}$; (c) $-\frac{\sqrt{2}}{2} + i\frac{5\sqrt{2}}{2}$; (d) $-16 - 16i$; (e) $-4\sqrt{2} - 2 + i\left(4\sqrt{2} - 2\sqrt{3}\right)$. **1.11** $\cos n\theta = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^m\binom{n}{2m}\cos^{n-2m}\theta\sin^{2m}\theta$; $\sin n\theta = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor}(-1)^m\binom{n}{2m+1}\cos^{n-2m-1}\theta\sin^{2m+1}\theta$; $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$; $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$; $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$; $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$. **1.12** z is real. **1.14** $\operatorname{Arg}(z) < 0$ unless $\operatorname{Arg}(z) = \pi$ in which case $\operatorname{Arg}(1/z) = \pi$. **1.15** $z = e^{i\theta}$, $-\pi < \theta \le \pi$. **1.16** $z = \frac{8}{5} - i\frac{9}{5}$, $|z| = \frac{\sqrt{145}}{5}$. **1.18** $\frac{263}{32}e^{-i\pi/2}$. **1.19** (a) $\pm\left(\frac{3\sqrt{2}}{2} + i\frac{3\sqrt{2}}{2}\right)$; (b) $2^{1/6}\cos\frac{\pi}{12} + i2^{1/6}\sin\frac{\pi}{12}$, $-2^{1/3} + i2^{-1/3}$, $2^{1/6}\cos\frac{17\pi}{12} + i2^{1/6}\sin\frac{17\pi}{12}$; (c) $32\cos\frac{n\pi}{8} + i32\sin\frac{n\pi}{8}$, where $n = 5, 7, 11, 15$; (d) $i2^{1/3}$, $-\frac{\sqrt{3}}{2^{2/3}} - i\frac{1}{2^{2/3}}$, $\frac{\sqrt{3}}{2^{2/3}} - i\frac{1}{2^{2/3}}$. **1.22** (b) (i) $\frac{i}{2}\left(-1 \pm \sqrt{5}\right)$, (ii) $-\frac{1}{2} \pm \left(\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{4}\right)$, (iii) i and $-2 - i$, (iv) $e^{in\pi/3}$ for $n = 1, 2, 4, 5$. **1.23** $(X, Y, Z) = \Phi(z) = \left(Kx, Ky, \frac{|z|^2-1}{|z|^2+1}\right)$, where $K = \frac{2}{|z|^2+1}$.

2.1 (a) $f_0(-27) = 3e^{i\pi/3}$, $f_1(-27) = -3$, $f_2(-27) = 3e^{i5\pi/3}$; (b) $f_0(-8i) = 2\sqrt{2}e^{-i\pi/6}$, $f_1(-8i) = 2\sqrt{2}\,i$, $f_2(-8i) = 2\sqrt{2}e^{i7\pi/6}$; (c) $z = 64e^{-i\pi/3}$, $k = 2$. **2.4** center: $(3 - 4i)/4$, radius: $5/4$. **2.5** normal: $A = \frac{2+3i}{\sqrt{13}}$, distance from origin: $|c| = \frac{1}{2\sqrt{13}}$. **2.15** left, bottom, and right boundaries of R map to the intervals $(-\infty, -1]$, $[-1, 1]$, and $[1, \infty)$ on the real axis, respectively; the positive imaginary axis maps to itself. **2.17** (a) $i\pi$; (b) $1 + i\frac{3\pi}{2}$; (c) $\frac{1}{2}\ln 2 + i\frac{7\pi}{4}$. **2.18** (b) the set of values for $n\log z$ is contained in the set of values for $\log(z^n)$; (c) equation is valid if $-\pi < n\operatorname{Arg}(z) \le \pi$ **2.19** (a) 1; (b) $2e^{\pi^2/3}e^{i(\ln(2)-1/3)}$; (c) $e^{-5\pi/6}e^{i5\ln 2\sqrt{3}}$. **2.21** (a) $k\pi i$, $k \in \mathbb{Z}$; (b) $i\left(\frac{\pi}{2} + k\pi\right)$, $k \in \mathbb{Z}$; (c) $i\pi\left(k + \frac{1}{4}\right)$, $k \in \mathbb{Z}$; (d) $2k\pi - i\ln(2 \pm \sqrt{3})$, $k \in \mathbb{Z}$; (e) $\frac{\pi}{2} + 2k\pi \pm 4i$, $k \in \mathbb{Z}$; (f) $\pi\left(k + \frac{1}{3}\right) - \frac{i}{2}\ln 2$, $k \in \mathbb{Z}$.

3.5 $\arg(w_2) - \arg(w_1) = \arg\left(\frac{w_2}{w_1}\right) \approx \arg\left(\frac{f'(z_0)\Delta z_2}{f'(z_0)\Delta z_1}\right) = \arg\left(\frac{\Delta z_2}{\Delta z_1}\right) = \arg(\Delta z_2) - \arg(\Delta z_1)$. **3.6** (a) $z = 1$; (b) everywhere except $z = 1$; (c) $x = y^2$; (d) $z = 0$ and $\theta = \frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$. **3.9** $R_r = \frac{1}{r}R\Theta_\theta$, $R\Theta_r = \frac{-1}{r}R_\theta$. **3.10** (a) analytic on all of $\mathbb{C}$; (b) differentiable on $x = -1$, not analytic anywhere. **3.11** $v = -r^2\cos(2\theta) + A$; $f'(z) = -2rie^{i\theta} = -2iz$. **3.14** (a) $v = -e^x\cos y + A$; (b) $v = \frac{1}{2}(y^2 - x^2) + y + x + A$; (c) $v(r, \theta) = \theta + A$. **3.16** $z = 0$ is a pole of order two, $z = i2\pi k$, $k \in \mathbb{Z}$, $k \neq 0$ are simple poles. **3.18** $z = \pm i$, $z = \pm i\sqrt{2}$, $z = \infty$.

4.1 $2i$. **4.2** (a) $4i - \ln 3$; (b) $1/z$ provided $\mathrm{Re}(z) > 0$. **4.4** $A = \frac{1}{2}(e^x \cos x + e^x \sin x - 1)$, $B = \frac{1}{2}(e^x \sin x - e^x \cos x + 1)$. **4.5** $f(x) = 2 + \sum_{k\in\mathbb{Z}} \frac{-2i}{(2k+1)\pi} e^{i((2k+1)\pi x)/L}$. **4.6** $f(x) = 3\pi - \sum_{k\in\mathbb{Z}} \frac{2}{(2k+1)^2\pi^3} e^{i((2k+1)\pi x)}$. **4.8** (a) $\frac{1}{2}(1 - \ln 2) + i\left(\frac{3\pi}{4} - 1\right)$; (b) $-1 - \frac{i}{2}$. **4.9** $-18\pi + i18(\pi - 3)$. **4.10** (a) $\frac{7}{3} - 5i$; (b) $\frac{7}{3} - i\frac{14}{3}$. **4.11** (a) $\frac{2}{3}(1 + i)$; (b) $\frac{1}{3} + i\frac{4}{3}$; (c) $3\pi - 2$; (d) $-\frac{8\sqrt{2}}{3}(1 - \sin(\pi/8))$ for principal branch, $\frac{8\sqrt{2}}{3}\cos(\pi/8)$ for branch with cut not crossing C. **4.13** (a) $\int_{C_1} f(z)\,dz = i\frac{3\pi}{2}$, $\int_{C_2} f(z)\,dz = -i\frac{\pi}{2}$, f is not analytic at $z = 0$ which is inside $C_1 - C_2$; (b) $-\frac{8}{3} + i\frac{8}{3}$ for both contours, f is analytic everywhere.

5.2 (a) $I = 1 - \cosh 2$; (b) 0; (c) $\ln(3) - i\pi$. **5.3** $\frac{1}{2}(-\ln(3) + i\pi)$. **5.5** (a) $z\,\mathrm{Log}\,z - z$; (b) $2i$; (c) $2 - 2\pi$. **5.6** (a) $-i\pi$; (b) $-\frac{27}{4} - 6\sqrt{3}i$. **5.7** radius greater than 1. **5.8** (a) $25\pi \sinh 4$; (b) $\frac{\pi}{4}(i - 1)$. **5.10** (a) C does not cross $z = t - i$, $t \le 4$; (b) $-\frac{\pi}{24}$. **5.13** Apply Liouville's theorem to $g = e^f$; yes, consider $g = e^{-f}$. **5.14** (a) $\frac{\pi}{8}$; (b) $\frac{2\pi}{3}$. **5.15** max $|f| = 21$ at $z = 3$, min $|f| = 0$ at $z = 0$. **5.16** max $|f| = \sqrt{(\ln 2)^2 + \left(\frac{3\pi}{2}\right)^2}$ at $z = -2i$, min $|f| = \frac{\pi}{2}$ at $z = i$. **5.17** max $u = 4\sqrt{3}$ at $(x, y) = (\sqrt{3}, -1)$, min $u = -4\sqrt{3}$ at $(x, y) = (-\sqrt{3}, -1)$. **5.19** (a) $6\pi i$; (b) $2\pi i$; (c) $-6\pi i$. **5.20** $6\pi i$. **5.21** one simple zero. **5.23** (a) 6π, 3 times; (b) -2π, -1 times; (c) 10π, 5 times.

6.3 (a) $|z| < e$, bounded; (b) $\left|\frac{z-i}{z+2}\right| < \pi$, not bounded. **6.4** uniform convergence on $x \ge 0$, diverges when $x < 0$. **6.6** $a_n(z - z_0)^n$. **6.7** (a) $\sum_{n=0}^{\infty} \frac{e}{n!}(z - 1)^n$, $|z| < \infty$; (b) $\sum_{n=0}^{\infty} \frac{e^4 + (-1)^n e^{-4}}{2n!}(z - 4)^n$, $|z - 4| < \infty$; (c) $-2\sin(2) + \sum_{n=1}^{\infty} \frac{(n+2i)e^{2i} + (n-2i)e^{-2i}}{2n!}(z - 2i)^n$, $|z - 2i| < \infty$; (d) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(z - \frac{\pi}{2}\right)^{4n+1}$, $\left|z - \frac{\pi}{2}\right| < \infty$; (e) $z + \frac{1}{3}z^2 + \frac{2}{15}z^3 + \cdots$, $|z| < \frac{\pi}{2}$; (f) $1 + \frac{1}{3}z^2 - \frac{1}{45}z^4 + \cdots$, $|z| < \pi$. **6.8** (a) $\sum_{n=0}^{\infty} \frac{e}{n!}z^{2n}$, $|z| < \infty$; (b) $\sum_{n=1}^{\infty}(-1)^{n+1}z^n$, $|z| < 1$; (c) $z + z^2$, $|z| < \infty$. **6.9** (a) $\frac{A+B}{2B} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(B-A)}{(2B)^{n+1}}(z - B)^n$; (b) $R = 2|B|$; (c) $-\frac{i10\pi(B-A)}{(2B)^5}$; (d) $i2\pi e^B\left(\frac{2B(A+B)+B-A}{(2B)^2}\right)$. **6.10** (a) R; (b) R^4; (c) $\sqrt{R}$. **6.11** $\sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}}$. **6.12** $\sum_{n=-1}^{\infty} \frac{(-1)^{n+1}}{4^{n+2}}(z - 1)^n, 0 < |z - 1| < 4$; $\sum_{n=2}^{\infty} \frac{(-1)^n 4^{n-2}}{(z-1)^n}, 4 < |z - 1|$. **6.14** (a) (i) $R = \sqrt{2}$, (ii) $R = \sqrt{5}$, (iii) $R = 1$; (b) $z_0 = 0$ on $0 < |z| < 1$, $z_0 = 5$ on $0 < |z - 5| < 5$ or on $5 < |z - 5| < 6$. **6.16** (a) pole order $\max(m, n)$; (b) pole order $m + n$; (c) pole order $m - n$ if $m > n$, else removable. **6.17** $f(z) = a_1 z + a_2 z^2 + a_3 z^3$. **6.18** (a) 5; (b) ∞; (c) pole order 4; (d) 3; (e) z_2 must exist, it is on $|z| = 3$; (f) $2 \le R_1 \le 8$; (g) $-\frac{5i}{2}$; (h) $-\frac{10\pi}{81}$.

7.1 (a) 12; (b) 0; (c) $-\pi$; (d) $\frac{1}{4\sqrt{\pi}}\left(-\sqrt{3} + i3\right)$. **7.3** yes. **7.4** (a) $\frac{\pi i}{12}$; (b) $8\pi i$; (c) $\frac{8\pi i}{\sqrt{3}} \log\left(2 - \sqrt{3}\right)$. **7.5** (a) $\frac{\pi}{13}(-27 + 5i)$; (b) 0. **7.6** (b) $-2\pi i$. **7.7** (a) $\frac{2\pi}{\sqrt{3}}$; (b) $\frac{2\pi}{\sqrt{11}}$; (c) $\frac{4\pi}{21\sqrt{21}}$; (d) $2\pi(3 - 2\sqrt{2})$. **7.8** (a) $\frac{(4-\sqrt{3})\pi}{13}$; (b) $\frac{3e^{-3a}(3a^2+3a+1)\pi}{8a^5}$; (c) $\frac{\pi}{2a}e^{-1/(a^{1/4}\sqrt{2})}\cos\left(\frac{1}{a^{1/4}\sqrt{2}}\right)$. **7.10** $\pi\,(\sin(1) - 2\sin(2))$.

8.2 not conformal at $z = \pm 1$ and $z = 0$. **8.4** $h(r, \theta) = T_1 + \frac{2(T_2-T_1)}{\pi}\,\mathrm{atan2}(1 - r^2, 2r\sin\theta)$. **8.5** $h(r, \theta) = T_1 + \frac{T_2-T_1}{\pi}\,\mathrm{atan2}\left((1 - r^2)\sin\theta, -(1 + r^2)\cos\theta\right)$. **8.7** $h(r, \theta) = \frac{R_2T_2-R_1T_1}{R_2-R_1} - \frac{2(T_2-T_1)R_1R_2\cos\theta}{(R_2-R_1)r}$.

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